1 Lambert's Perspective Geometry

by

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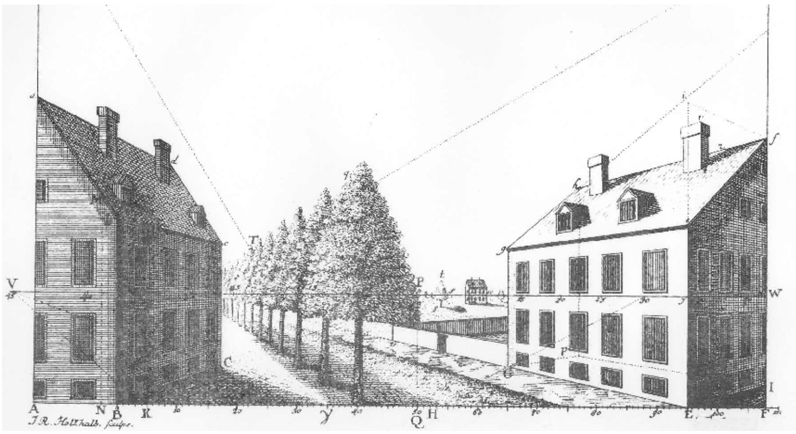
Lambert's approach to perspective constructions

In the history of mathematics, Johann Heinrich Lambert is best remembered for making isolated clever observations rather than for having created a theory. Thus, it is well known that he proved that the number π is irrational and that he had remarkable thoughts about the possibility of a geometry in which Euclid's postulate five is not valid. It is considerably less known that he actually did work out a complete mathematical theory, namely in the field of perspective (for a longer presentation of Lambert's works, see Scriba 1973; and for his work on perspective, see Laurent 1987 and Andersen 2006, chapter XII). Having acquainted himself with the contributions by his predecessors, he took an innovative approach to the theory of perspective, hitting on the idea to consider a picture plane as possessing its own geometry. He called this new discipline perspective geometry. Lambert presented his ideas in a book which appeared simultaneously in French and German under slightly different titles, namely (in English translations) *Perspective freed from the nuisance of a geometrical plane* and *Free perspective, or instruction in making any perspective drawing of one's own choice without a plan* – hereafter referred to as the *Freye Perspektive* (Lambert 1759_1 and 1759_2). Lambert did not stop thinking about perspective after having finished his book, but took up new and ever more challenging questions. Some of his answers were published when the German edition of his book was reprinted (Lambert 1774).

Before Lambert, it had been customary to base the construction of the perspective image of an object upon a plan and elevation of the object. As Lambert stressed in the titles of the *Freye Perspektive*, his new approach was to perform constructions directly in the picture plane – it was in that sense it got its own geometry with its own construction rules.

In showing how to perform constructions in perspective geometry Lambert began with horizontal polygons, for which he applied two primary concepts: a vanishing point and a measure point. To construct these points, he made use of a so-called "angle scale". The latter is a scale constructed on the horizon by designating by v the vanishing point of lines making the angle v with the perpendicular to the picture plane. Vanishing points had played an important role in the theory of perspective since 1600; measure points and angle scales were not new either, but within Lambert's perspective geometry these geometrical objects were given a much more functional role than previously. To construct the image of traditional objects, Lambert also needed a method for constructing perspective heights. However, perspective heights were unproblematic as they had always been constructed directly in the picture plane.



1Lambert's straightforward example. Lambert 1759, figure 14.

For constructing the perspective image of a composition as the one shown in figure 1, Lambert's new approach was very helpful. His technique was straightforward to apply, because it only needed the construction of an angle scale. The elegance of his method was noticed by a number of authors and practitioners. Thus, the German mathematician Wencelaus Johann Gustav Karsten presented Lambert's method in 1775 – even copying his illustration shown in figure 1 (Karsten 1775, figure 68). Furthermore, as Sabine Seibel has documented, in the years around 1800, a group 1connected to the academies of art in Berlin and Dresden advocated basing the teaching of perspective on Lambert's ideas (Siebel 1999, 23, 91, 96, 169–171). Siebel has also pointed to some early nineteenth-century German authors who, to varying degrees, were inspired by Lambert.

1With his *Freye Perspektive,* Lambert aimed at reaching those people who made perspective constructions as part of their profession – among them were painters, graphic designers, and architects. For this purpose, he found it proper to present his ideas as a collection of rules with arguments for some but not for all. This was contrary to the way mathematicians preceding him had styled their writings on perspective. They had kept to the formal way of presenting geometry that involved theorems, problems, and proofs. As mentioned, Lambert managed to some extent to attract the attention of the practitioners of perspective – but only to some extent. When the reader had learned to perform basic constructions, he had only read about a third of the *Freye Perspektive*, and most readers would presumably stop then. In the rest of his book, Lambert showed that he had much higher ambitions than making compositions like the houses in figure 1. Indeed, it was Lambert's goal to teach much more general constructions, such as how to construct objects consisting of oblique sides, and to construct water jets, rainbows, and starry skies.

For mathematicians Lambert's solutions of the more advanced problems are not too difficult, but it is intricate to keep track of all the steps and arguments involved. Thus, in working with some of the complicated examples Lambert made many clever observations which he applied to simplify his constructions. For instance, when constructing reflections, he applied the fact that several points that are different in the three-dimensional space may coincide in the perspective plane (they being situated on the same line through the eye point). However, Lambert often found it unnecessary to reveal his observations, and it therefore requires some contemplation to understand the constructions he prescribed. For the non-mathematical readers, my impression is, as noted, that most of them would give up. Indeed, if one really wants to carry out – and not only admire – some of Lambert's advanced and theoretically elegant constructions, it would be so work-consuming that conventional solutions might be easier.

Perspective geometry applied to other branches of geometry

After having solved all the traditional kinds of perspective constructions – and many more – inside his perspective geometry, Lambert proceeded to apply his new discipline to solving problems in ordinary geometry.

One of Lambert's applications concerned parallel projections. Conceiving of a parallel projection as a perspective projection with its eye point at infinity, Lambert developed a method for direct constructions of the images of angles and line segments under a parallel projection. In this connection he proved a precursor of Pohlke's theorem, first formulated by Karl Pohlke in 1853 (Schwarz 1864).

While performing perspective constructions, Lambert noticed that the ruler is used much more frequently than the compass. This observation made him, as he explained in a letter to Karsten, approach "geometry again in order to see – under the guidance of the laws of perspective – how far one can go in elementary geometry if one is only allowed to use a ruler" (1Lambert *Briefe*, vol. 4, 325). In doing so, he was led to a result that is a precursor of the theorem later called the Steiner Circle Theorem or the Poncelet-Steiner Theorem that first was published by Jean Victor Poncelet and later by Jakob Steiner (Poncelet 1822, Steiner 1833).

The impact of Lambert's work

As mentioned, some of Lambert's elementary constructions were taken up and made Johann Erdmann Hummel write:

> We have to thank the acumen of our Lambert for the foundation ... of the theory that includes all that which can be called perspective drawing; any attempt to think out something more perfect would be futile (Hummel 1824, V).

Hummel was right: Lambert's creation was a final touch to a long process of coming to understand the geometry of perspective. His solution was so comprehensive that in the theory of perspective *per se*, little work was left to do after Lambert. Still, Lambert's theoretical impact seems to have been negligible. For instance, his work went largely unnoticed by mathematicians and, hence, the developments that took place within descriptive and projective geometry were not influenced by Lambert. One of the reasons that mathematicians did not read Lambert might be his choice of a non-mathematical style for presenting his ideas. His style made Karsten describe the *Freye Perspektive* as *das schöne Handbuch* – the lovely handbook – rather than a mathematical book.

Although Lambert did not influence the subsequent development of important mathematical disciplines, his ideas deserve to be remembered. Jean Victor Poncelet was one of the first to call attention to a clever solution by Lambert (Poncelet 1822, §199). Poncelet returned to France from Russia in 1814, after having created projective geometry while in captivity as a prisoner of war. Before he published his new ideas in 1822, he read an impressive amount of geometrical literature – presumably to find problems that he could solve elegantly with the help of his projective arguments. Most likely, it was in this connection that he became aware of Lambert's work on perspective.

References

1 **Andersen, Kirsti**

2006 *The Geometry of an Art. The History of the Mathematical Theory of Perspective from Alberti to Monge*, New York.

Hummel, Johann Erdmann

1824 *Die freie Perspektive, erlaütert durch praktische Aufgaben und Beispiele, hauptsächlich für Maler und Architekten*, 2 vols., Berlin 1824 and 1825.

Karsten, Wenceslaus Johann Gustav

1775 "*Die Perspectiv*" in *Lehrbegrif der gesamten Mathematik*, Greifswald, vol. 7, 110–928.

Lambert, Johann Heinrich

Briefe Joh. Heinrich Lamberts deutscher gelehrter Briefwechsel, ed. Johann Bernoulli, 5. vols., Berlin, 1781–1787.

1759_1 *La perspective affranchie de l'embaras du plan géométral*, Zürich. Facsimile reprints Paris 1977 and Alburgh 1987. German edition below.

1759_2 *Die freye Perspektive, oder Anweisung jeden perspektivischen Aufriss von freyen Stücken und ohne Grundriss zu verfertigen*, Zürich. Second edition as part of Lambert 1774. Reprinted in Lambert 1943, 192–301.

1774 *Die freye Perspective ... mit Anmerkungen und Zusätzen vermehrt*, Zürich. Reprinted in Lambert 1943, 309–380. French translation of the *Anmerkungen* in Laurent 1987, 193–284.

1943 *Johann Heinrich Lambert, Schriften zur Perspektive*, ed. Max Steck, Berlin.

Laurent, Roger

1987 *La place de J.-H Lambert (1728–1777) dans l'histoire de la perspective. Suivi de la version intégrale, inédite en français des Notes et additions à la perspective affranchie ... traduction de Jeanne Peiffer*, Paris.

Poncelet, Jean Victor

1822 *Traité des propriétés projectives des figures*, Paris. Second edition Paris 1865.

Scriba, Christoph J.

1973 "Lambert", *Dictionary of Scientific Biography*, ed. Charles Coulston Gillispie, New York, vol. 7, 595–600.

Siebel, Sabine

1999 *Die Ausbildung in der Perspektive an den deutschen Kunstakademien um 1800*, Dissertation zur Erlangung der Würde des Doktors der Philosophie der Universität Hamburg, Hamburg.

Schwarz, Hermann A.

1864 "Elementarer Beweis der *Pohlke*schen Fundamentalsatzes der Axonometrie", *Journal für die reine und angewandte Mathematik*, vol. 63, 309–314.

Steiner, Jakob

1833 *Die geometrischen Konstructionen ausgeführt mittelst der geraden Linie und eines festen Kreises*, Berlin.