Descartes' *Géométrie* and history of geometry

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1. In a booklet, published some 45 years ago, J.Vuillemin made clear that there is a central notion in *La géométrie* (henceforth *G*): the concept of *genus*. Moreover Vuillemin, gave sufficient indications to understand that Descartes (henceforth D) treats Fermat contemptuously because he disagrees with his conception of the continuum.

These 3 topics : 1)Why this concept of genus ? (wrongly translated by « classes »), 2)Why this distaste for Fermat's mathematics?, 3)What is D's concept of continuum ?, seem typical of the questions which should concern historians of mathematics. But more recent works about *G* are mute on these issues. Either they don't even mention the presence of the concept of genus or, when they do, they merely say this is an obsolete concept because it does not fit in the algebraic classification of curves (that we own to D !).

Genus is a two-fold concept. On one side it provides a *measure of complexity* of the *finite constructions* allowed in *G*. On another side, it corresponds to the concept of physical dimension invented by D (and forgotten by Newton). I sketch below the realm of ideas one has to consider to understand D's project in *G*, namely the determination of the parts of mathematics required by physics. such determination beeing subordinated to D's epistemological principles : intuitionism, dualism and mechanicism.

2. D is claimed to be the father of modern science and modern philosophy. This could rise 2 questions : 1) What was new ? 2) What came first ? science or philosophy ? Answers come out from the comparaison of *Regulae* (henceforth *R*) with D's other works from 1629 on. In *R* we already find all the mathematical content of *G*. This content is obtained by giving up two principles instituted by Aristotle

1°) Law of homogeneity : One cannot write in a same expression quantities of different orders ; e.g. $a+b^2=c^3$. This law dominates mathematics from Antiquity up to Fermat

2°) Principle of incommunicability of genus : One cannot, e.g., prove an arithmetical proposition by geometrical means (and vice versa). Because each genus has its own principle (unity for arithmetic and magnitude for geometry).

Giving up this last principle, D neutralizes the opposition between arithmetic and geometry. *This led him in a direction opposed to the dominating present day conception of mathematics* which, by the medium of analytic geometry, reduced geometrical knowledge to (transfinite) arithmetic. This is a 1st element of his intuitionism.

D's new anti-aristotelician principles are both a consequence of the *theory of proportions*. This theory allows D to define the 5 rational operations in the same way as he does in the first pages of *G*. It must be added that the theory of proportions contains an inherent order figured by the fact that the numbers 3, 6, 12, 24,…are in a continued proportion.

The whole *R*'s project, namely the construction of a « science générale qui explique tout ce qu'il est possible de rechercher touchant l'ordre et la mesure, sans assignation à quelque matière particulière » seemed at hand. But D gave up this project. To break with the old mathematical tradition was not enough. Epistemology which was still aristotelian had to be thrown out, like old maths had been.

Hence, we have an answer to the 2nd question above :*D just wanted to be the father of modern science by creating a new Mathesis Universalis. But this demonstrated impossible. He had to be the father of both modern science and modern philosophy to really be, as he was, the father of modern science*.

3. Mathematical content of *G* and of *R* are equal. But *G*, which is still the science of order and mesure, does not have anymore the function that Universal Mathematics had in *R*. Hence, to understand what was *G* for D, one has to turn towards metaphysics.

Historians of mathematics frequently overlooked the fact that *G* is not « pure » geometry but physical geometry. More precisely, it is exactly the mathematics needed by D'physics. Among the evidences supporting this claim, let's mention that (i) 30% of Book II of *G* is dedicated to optics, (ii) in *G*, D explicitely stated that he wants to *construct*. all the curves « *qui sont dans la nature* » (Smith's translation omits this phrase), (iii) D, from *R* on, repeatedly says he is not anymore interested in mathematical problems, (iv) *G* is a part of a larger book named **«** *Discours de la méthode pour bien conduire sa raison, et chercher la vérité dans les sciences. Plus la Dioptrique, les Météores et la Géométrie qui sont des essais de cette méthode* » (henceforth *Discours*).

Two main reasons led D to turn upside down the function of *G* in the foundations of mathematical physics.

1) An insatisfying doctrine of faculties.

Order, as it works in the ordering of proportions is intellectual ; it depends on *understanding* and cannot be conciliated with the natural order of beings in genus and species of aristotelism. Moreover, how is the 1st element of order given to us ? since, with the neutralisation of the opposition between arithmetic and geometry, the unity, which is the principle of arithmetic, has disappeared ?

In *R* we read **«** …if the understanding proposes to examine something that can be referred to the body, we must form the idea of that thing as distinctly as possible in the imagination ; and in order to effect this with greater ease, the thing itself which this idea is to represented must be exhibited to the external senses**. »** In other words, application of maths to the knowledge of the bodies (to physics), is subordinated to imagination which, in its turn, depends of sensation. I can apply the equation of the circle to describe the shape of the Moon because my senses (vision) give to my mind the representation of roundness, according to the following scheme :

Sensation → imagination → understanding, (where the arrow reads « is needed by »)

D gave up this aristotelian scheme because (i) « our senses deceive us » and (ii) imagination has inherent limits (I cannot imagine a chiliogone). Therefore, the essence of coporeal things (objects of geometrical reasoning) must come from elsewhere than from sensation : they are formed by the understanding.

2) Theory of vision.

Kepler noted that, as a consequence of the substitution of light rays to visual rays, images of the sawn things are formed on the retina. But he confessed to be unable to explain how these images could propagated along the optical nerve. D gave the solution in *La dioptrique* : the optical nerve works like a *transducer* converting the mechanical pression of light rays on the eye in an other kind of signals propagating along the nerve up to epiphysis (meeting point of the body and the soul). Hence, « c'est l'âme qui sent» (not the body).

Two problems must be solved in order « to arrive at any certainty in the sciences » : to prove that purely intellectual ideas can apply to the corporeal things outside of the mind and, more awkward :since I am aware of the existence of bodies by my deceiving senses, even these existences must be proved.

4. In this new context *G* is (i) the restricted part of the understanding which could be delegated to the imagination (ii) the most faithfull way *to picture* what the Method is. But *G* is

still the science of order and measure when this science is concerned by both order *and* measure. As such it opposes to three possibilities

(a) *a very limited science of measure without orde*r, described in Book I dedicated to geometrical curves constructible with ruler and compass.

Limited, because, in Book II, D shows that these constructions are just those of 1st genus. Physics needs more than that.

Without order : if a curve is given, one cannot recover how many times, and in what order, ruler and compass were used to construct it. There is a big novelty in *G* wrt *R* : occurrences of the gliding squares in Books II and III. The parts of the gliding squares are joined together (as D's reasons are and as the essays of *Discours* are). Ruler and compass are obviously disjointed. When we construct a curve with ruler and compass we do not do it by a continuous motion, but by a sequence of interrupted ones. This gives D's meaning of « continuous » (= not interrupted). With the gliding squares the previous indetermination disappears : *recoverability of operations entails reversibility of analysis and synthesis* (and show that the distinction between analytic and synthetic geometry is irrelevant in *G*). The gliding squares act as *a finite abstract analogical automata*. Complexity of the *functions* computed by this device is measured by order of their genus (i.e . number of proportions).

(b) *Pseudo-science of measure in disorder*. When a curve is (i) constructed by more than one machine (instrument) and (ii) not constructible by the gliding squares, D calls it *mechanical*. By definition, mechanical curves are excluded from *G*.

In his letters, D elucidates the properties of several transcendent curves. The logarithmic spiral is one of them. This case is of special interest because it throws some light on D's finitism. Vuillemin observes that it cuts an indefinite number of gliding squares in each of their joining points. To determine its locus requires a computation involving an unlimited number of proportions.

Rejection of transcendental curves is consistent with D's epistemological decision to bannish the study of sensation. Sensations can be measured but, not satisfying the law of additivity, they have no extensive magnitude (functionally computable).

In short, for D, mechanical constructions, transfinite reasoning and intensive magnitudes are in some way, equivalent things. They define what must be kept outside of science of order and

measure. This rejection of large parts of analytical geometry is a 2nd element of D's intuitionism.

(c) Take off measure from the science of order and measure. It remains the science of pure order: metaphysics. It is unavoidable (Cf. the 2 problems closing sec.**3**). It concerns exclusively the *understanding*. It shows up a precise, and original, doctrine about the relationships between the infinite and the finite which rests on the distinctions between the *finite*, the *indefinite* and the *infinite*. D claims to be the 1st to put «de la distinction entre l'*indéfini* et l'*infini* ».

Example of metaphysical deduction : *The infinite is a full and real being. The finite is the negation of the infinite. The negation of a being is not a being. The indefinite is the negation of the finite. The negation of a non-being (non être) is not a being*.

Hence : (i) the rule of double negation is rejected (3rd sign of intuitionism) and negation of the finite is not the infinite,

(ii) Since « The truths of metaphysics are more certain than the truth of mathematics because they are easier to prove », we conclude that the rejection of the principle of excluded middle (and its correlate the *reductio ad absurdum*) applies to the determination of the content of *G* (as it has been checked above on the case of transcendental curves).

5.After D's intuitionism and mechanism, let's see his dualism. For D the subject matter of mathematical physics is an ordered pair <Man, World>. The World is made of two things : isotropic light and inhomogeneous matter. In such a conception, metrical description (given by *G*) is provided by properties of straight light rays. Light rays act mechanically (rigidly) on Man's eyes. This action propagates up to the soul which computes, with *G*, extensive and instantaneous magnitudes of what it sees. Ten years before D completed his doctrine, Kepler was still explaining motions of planets by properties of their souls. D reserved the possession of a soul to men. This made him the father of modern philosophy. To invent mathematical physics as we know it today, Newton transformed this dualism, He took off mathematics objects from Man's soul and filled the World with them. He letf thus to future physicists a riddle they have not yet overcome.