The Early Modern Tradition of Geometrical Problem Solving

Summary of the lecture by Henk J.M. Bos

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The title refers to a recognizable field of mathematical activity during the early modern period. The peak of this activity was between the publication, in 1588, of the Latin translation of Pappus' *Collectio*, and the publication of Descartes' *Géométrie* in 1637. The geometrical problems were mostly of classical Greek origin; they were construction problems, often constructible by straight lines and circles ("ruler and compass"), but also, and importantly, problems which could not be so constructed.

During the indicated period geometrical problem solving faced two methodological issues:

1. How to construct "beyond" straight lines and circles.

2. How to find, in a methodical way, the appropriate construction for a given geometrical problem.

Pappus' *Collectio* provided the classification and the terminology for the discourse on the first issue. The same source provided challenging indications about a classical method as envisaged in the second issue, called 'analysis,' but incompletely documented in the extant sources. The issue was further complicated by the growing belief that the new methods of algebra might provide and alternative method to find solutions, that is, constructions of geometrical problems.

The two methodological issues made geometrical problem solving a challenging field in mathematics, attracting many geometers. The most productive among these were Viète, van Roomen, Anderson, van Ceulen, Snellius, Ghetaldi (each with 4 or more relevant publications); during the period 1588-1637 at least 34 authors together produced more than 80 publications on geometrical problem solving.

However, if one measures the score of this activity by the number of problems solved which are still generally remembered, the result is hardly impressive. The only convincing candidates are Apollonius' problem (solved by Viète, the classical Greek solution remains unknown), and the problem 'in n lines,' (solved by Descartes, the classical partial solution is also lost). The tradition did not produce geometrical theorems of lasting status in the geometrical corpus. Thus it is not surprising that the tradition itself has not acquired a noticeable place in our historical picture of mathematics.

Yet the field played a very important role in the development of mathematics, because it provided a context for:

1. The early modern merging of algebra and geometry;
2. Viète's abstract algebra (letter algebra);
3. Descartes' and Fermat's analytic geometry;
4. Effective and influential discussions on mathematical exactness.

In the remainder of the lecture I discussed three topics to illustrate the two methodological issues within the early modern tradition of geometrical problem solving and their role in the endeavours to merge algebra and geometry (leading eventually to analytic geometry). The first was a triangle problem, apparently first proposed by Regiomontanus. The problem was: Given line segments c, h, d, e, to construct a triangle ABC, with base c, height h, and ratio $a : b$ of its sides equal to $d : e$. Regiomontanus could not solve it by ruler and compass and only provided an algebraic method to calculate the lengths of sides a and b for given numerical values of c, h, d, e.

Because Regiomontanus had explicitly presented this result as a failure, the problem acquired considerable fame, and the reactions by various mathematicians give a good insight in the discussions about the two methodological issues mentioned above.

The second topic was Viète's proof wthat all solid problems, by which he meant all problems that can be reduced to an algebraic equation of degree 3 or 4 (but not 2), can be constructed if one assumes that constructions are available for two of the three classical problems, namely the trisection of any given angle and the determination of two mean proportionals between any two given line segments. He also showed that these problems could indeed be constructed if one allowed a particular procedure called a 'neusis construction.' An impressive result because it provided an ordering of a large class of geometrical problems in terms of their constructions. However, several steps of Viète's argument were algebraic without geometrical translation, so that he proved an implicit constructibility rather than an explicit construction.

The third topic was Descartes' proof, in the *Géométrie*, that every problem reducible to a third- or fourth-degree equation can be constructed by the intersection of a parabola and a circle. The proof consisted of an explicit construction, derived algebraically (presumably by an indeterminate coefficients argument, which he left unexplained) but proved geometrically.

The lecture was meant to show that the early modern tradition of geometrical problem solving deserves recognition as an important phase in the development of mathematics.

Although its methodological difficulties, as well as its results, are now largely forgotten, the challenges it presented to mathematicians and the processes of development occasioned by these challenges, can be re-appreciated and understood.

Much of the material discussed in the lecture can be found in my monograph *Redefining geometrical exactness : Descartes' transformation of the early modern concept of construction*, New York, Springer, 2001.