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Hermann Grassmann and the concept of extensive magnitude

Summary

The paper investigates Grassmann's distinction between the concept of continuous magnitude and the concept of discrete quantity in the context of his foundational program, which amounts to separate foundations for geometry and arithmetics. Grassmann's philosophical frame is examined in relation to the concept of extensive magnitude, which constitutes the object of a new mathematical branch: extension theory. References include both the first (1844/1878) and the second edition (1862) of *Ausdehnungslehre as well as Geometrische Analyse.* Relevant differences concerning the foundation of extension theory are explained as a result of distinct purposes of the author: a philosophical foundation of the new theory versus a mathematical exposition of its main results.

The concept of magnitude, as it appears in the first edition, is showed to be connected to the criticism of the adequacy of the definition of mathematics as 'Grössenlehre' and to the effort to found geometry (or rather its mathematical counterpart, extension theory) independently of arithmetics. According to Grassmann, the German term 'Grösse' is inadequate to the task of denoting all mathematical objects. Firstly, 'Grösse' denotes something that changes in a continuous way, which Grassmann proves by a linguistic comparison of the words 'vermehren' and 'vergrössern', whereas some mathematical objects are discrete (e.g. numbers). Secondly, 'Grösse' fails to express what is essential to mathematical objects, that is to say, that they are generated according to a rule. Being determined by a generating act rather than being considered as given, mathematical objects are called 'Denkformen' or simply 'Formen'. However, Grassmann does not strive to find a genus including all kinds of mathematical objects as species: the property of being generated from a generating element according to a definite rule is still an abstract, empty definition, unless it is filled with some particular content. Mathematical objects are indeed determined by the nature of the generating act, which might be continuous or discrete, by the manner in which the generating element is conceived (e.g. as equal or as different), by some substitutional conditions (equality relations), and finally by the kinds of connection among objects (addition, multiplication, raising to a power) and their properties (commutative, associative, distributive, invertible, …).

Opposite to the first edition, the term 'Grösse' occurs in the second edition of *Ausdehnungslehre* as a general term, applying both to continuous and to discrete mathematical objects. However, this terminological shift does not alter Grassmann's main issue that one should distinguish between a general term that applies to all mathematical objects and a particular one, which applies only to continuous objects.

In the first edition, the concept of an extensive magnitude is defined as the class of extensive formations that are generated according to the same law by means of equal variations, for an extensive formation is conceived of as the collection of all elements into which the generating element is transformed by a continuous variation. An extensive magnitude is thus defined in terms of the concept of variation, which is assumed as a primitive one. The application of extension theory to space provides a clarificatory exemple: a variation of the generating element might be understood as the movement or change in position of one point thus generating a line segment; two generating elements that vary according to the same law might be seen as two points moving in the same direction. Of course, one might charge Grassmann with a lack of rigor, for he defines the concept of extensive magnitude by means of an intuition-laden concept, but one should also remark that he just failed to withdraw from the mainstream of contemporary geometry. Attention should rather be paid to the fact that Grassmann's definition may result from his foundational program: as he plans to build extension theory independently of other mathematical branches, he does not assume any arithmetical issue, not even real numbers.

In the second edition, Grassmann abandons the foundational perspective for fear that it might hinder an adequate understanding of his work. Still, he doesn't deny that a separate foundation of extension theory should be preferred. On the one hand, an extensive magnitude is defined as a polynomial expression determined by a system of units and derivation numbers relative to it. Since the continuous system of real numbers is assumed, no independent foundation for geometry can be laid down. On the other hand, Grassmann still claims that the philosophical exposition is more adequate, because it does not present extensive magnitudes as given but rather explains the process of their generation, and it does not define them on the basis of extraneous elements such as numerical coordinates.

The main topic discussed in this paper concerns the understanding of Grassmann's goals, as he founds extension theory (and geometry as an application of the former) without analytical coordinates. It is claimed that Grassmann has four related but distinct foundational goals: 1. to reflect the constructing movement of thought that determines relations and connections among forms, 2. to provide a scientific exposition of the theory, 3. to mark the difference between discrete numbers and continuous dimensional magnitudes, 4. to provide independent foundations for geometry and arithmetics. To this end, various quotations from Grassmann's works are examined. Particular attention is given to the relation between the numbers that are assumed as components of units and the numbers that measure extensive magnitudes, which are defined by means of outer multiplication. The assertion that the arithmetical product might be considered as a special kind of outer multiplication among extensive magnitudes is further discussed and explained with relation to Grassmann's 'geometrical purism'. The latter is exemplified by the criticism of the modern use of measures in the application of proportion theory to the study of similitude. The refusal to relegate the study of surface area and similarity to measurement is analysed in connection with the relation between algebra and geometry, and is presented as an exemple of Grassmann's aim at generality, which yields to a unified and uniform treatment of the geometrical theorems appearing in Books II and IV of Euclid's *Elements*.

In conclusion, an application to the theory of colours is briefly sketched and a short remark is made on the relation between Grassmann’s concept of an extensive magnitude and the concept of measurable quantity as it occurs in modern representational theories.