Hessenberg's work on projective geometry.
(Karin Reich)

The rise of geometry during the 19th century was closely connected with the development of the technical colleges. One of the new branches that was initialized was projective geometry which was founded by Poncelet, Steiner, Plücker, von Staudt and so on. As far as its application is concerned, projective geometry played a major role for example in graphical statics (Culman, Cremona, ...).

1. David Hilbert's Foundations (1899)

There is no doubt that Hilbert's "Foundations of Geometry" (1899) were a new peak in geometry and a turning point within mathematics. The axiomatical method propagated by Hilbert dominated sooner or later not only geometry, but all mathematical disciplines.
His "Foundations" contained 7 chapters: 1. The five groups of axioms, 2. The compatibility and mutual independence of the axioms, 3. The theory of proportion, 4. The theory of plane areas, 5. Desargues's theorem, 6. Pascal's theorem, 7. Geometrical constructions based upon the axioms I-V. In § 23 Hilbert showed the impossibility of proving Desargues's theorem in the plane without the aid of the axioms of congruence. After that he introduced a kind of segment arithmetic based on Desargues's theorem with the aid of the congruence axioms (§ 24). After his explanations of the Pascal's theorem he delivered the proof of any theorem relating to points of intersection: "Every proposition relating to points of intersection in a plane has necessarily the following form: "Select, first of all, an arbitrary system of points and straight lines satisfying respectively the condition that certain ones of these points are situated on certain ones of the straight lines. If, in some known manner, we construct the straight lines joining the given points and determine the points of intersection of the given lines, we shall obtain finally a definit system of three straight lines, of which our proposition asserts that they all pass through the same point" (§ 35).
The result is: "In order to show that, for the new algebra of segments, the expression $A(p_1 , p_2 ,..., p_n)$ vanishes identically, it is sufficient to apply the theorems of Pascal and Desargues. Consequently we see that: Every proposition relative to points of intersection in the geometry in question must always, by the aid of suitably constructed auxiliary points and straight lines, turn out to be a combination of the theorems of Pascal and Desargues. Hence for the proof of the validity of a theorem relating to points of intersection, we need not have resource to the theorems of congruence."

1.1. Foundations 1909, third edition, and onwards

From the third edition in 1909 onwards this result is accompanied ba the following footnote: "Hessenberg was the first who recognised that the Desargues theoreom is a consequence of the Pascal theorem. This can be derived without using the axioms of congruence and continuity. By means of this result it follows, as Hessenberg had indicated, the remarkable theorem: every point of intersection theorem can be proved on the basis of the Pascal theorem only, it is not necessary to make use either of the axioms of congruence nor of the axioms of continuity."

1.2. Foundations 1956, eighth edition, and onwards

After Hilbert's death in 1943 Paul Bernays became the editor of Hilbert's Foundations. Now this whole § 35 was changed, the footnote disappeared and there were two theorems instead:

"Desargues' Theorem can be proved from Pascal's Theorem with the aid of the Axioms I, 1-3, II and IV alone and hence without the use of the congruence and the continuity axioms". This theorem which is called a very important result, is proved thoroughly, that means, Hessenberg's proof is repeated. The § 35 ends with the following consequence: "Every pure point of intersection theorem that holds in a plane geometry in which Axioms I, 1-3, II, IV and Pascal's Theorem are valid takes, through the construction of suitable auxiliary points and lines, the form of a combination of a finite number of Pascalian configurations. In proving the point of intersection theorem with the aid of Pascal's Theorem it is then no longer necessary to revert to the congruence and continuity axioms".

2. Gerhard Hessenberg

Born in Frankfurt in 1874, Hessenberg studied in Strassburg and in Berlin. His thesis was written under the direction of J.Weingarten and J.Knoblauch and concerned differential geometry: Über die Invarianten linearer und quadratischer binärer Differentialformen und ihre Anwendung auf die Deformation der Flächen (1899). In 1903 he became lecturer for mathematics at the "Militärtechnische Akademie" in Berlin. In 1907 he started his career as professor of mathematics in Bonn, 1910 in Breslau (Wroclaw), 1919 in Tübingen and 1925 in Berlin where he died in the same year. He had rejected a position in Königsberg in 1919, a position in Berlin in the same year and a position in Leipzig in 1921. Indeed, Hessenberg belonged to the top geometers of his time.

2.1. Hessenberg's contribution to projective geometry

It was only during the interval 1901-1905 that Hessenberg contributed to projective geometry, he published 7 papers on this subject.

Preliminaries.

He finished his first paper in may 1901: Über Beweise von Schnittpunktsätzen (About proofs of point of intersection theorems). Here he presented several examples of point of intersection theorems which could be derived either by the only use of the Desargues theorem or by the only use of the Pascal theorem. In August of the same year he published two further articles, one about "Desargues's theorem and Central Collineation" and one about "A geometrical calculus". Both articles as well as his 1903 published paper "On projective geometry" were important steps in the direction of his main contribution.

April 1905: Proof of the Desargues' Theorem on the Basis of Pascal's Theorem.

The main point of Hessenberg's proof is that the Desargues configuration can be produced by a threefold application of the
Pascal configuration. The proof was given at first for a special case, i.e. the affine case and afterwards for the general case.

This was indeed a highly unexpected result, the Desargues theorem was not independent from the Pascal theorem, but it followed out of the Pascal theorem. Hessenberg's proof was highly respected and is still in use today.

Elliptic and spherical geometry.

In the same year Hessenberg published to further papers where he adopted his new results to the elliptic and to the spherical geometry.

2.2. Textbooks.

Hessenberg had published textbooks which were based on his lectures on descriptive and analytical geometry (1918/9 and 1922). After his death Wilhelm Schwan published

Hessenberg's "Foundations of Geometry" in 1930. Main subjects were the fundamental theorem in projective geometry, the Pascal theorem and the related Desargues theorem. It was proved that the fundamental theorem can be reduced to the Pascal theorem.

3. Award and honour.

As far as projective geometry is concerned, Hessenberg had investigated in a quite narrow field. All his publications had one major aim: the dependence of the Desargues theorem from the Pascal theorem and its consequences. And this result was and has to be well respected: in 1910 Hessenberg was decorated with the Leibniz' medal from the Prussian Academy of Sciences. At first he should have been honoured with the Steiner award, but this unfortunately had been given to Gaston Darboux. It should be mentioned that Hilbert himself had won the Steiner award in 1900. The report of Hessenberg's bestowment of the Leiniz medal is due to H.A.Schwarz. And, there still exist several letters which Hessenberg had addressed to Hilbert in 1905.