Non-euclidean Geometry from the Point of View of A. Cayley and F. Klein

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Nicholas Lobatschewsky and Janos Bolyai are the founders of non-euclidean geometry. In the first half of the 19th century they developed an harmonious geometrical structure, having apparently no inherent logical contradictions. Lobatschewsky called this structure “imaginary geometry”, Bolyai spoke of “absolute science of space”. Today we call this geometry “hyperbolic geometry”. However, the first systematic answer to the question: What is non-euclidean geometry? gave Arthur Cayley and Felix Klein. In the first part I explain the point of view of Cayley and Klein. In the second part I give a modern reconstruction of this conception of metrical geometry. Both parts were worked out in collaboration with Rolf Struve.

I. The work of Cayley and Klein

The central idea of Cayley and Klein was anticipated by Edmond Laguerre. In a note published in the year 1853 Laguerre made the following observation: The euclidean value of the angle made by two lines a and b is equal to 1/ 2i times the cross-ratio of a, b and the two isotropic lines through the point of intersection of a and b. Remarkable is firstly the use of complex numbers – more precisely: the extension of the real projective plane to the complex one – and secondly the definition of a metric concept (the measure of angle) by help of a concept from incidence geometry (cross-ratio).

In 1859 Cayley published a memoir which became famous in the history of geometry. The quintessence of this memoir was the insight that metrical properties are not properties of a figure per se but of the figure in relation to an absolute, a curve of second order in the complex projective plane.

Cayley posed the problem to define the euclidean measures of angles and of distances of points within incidence geometry. The idea of Cayley was first to develop such formulas with respect to a *non-degenerated quadric* and second to regard a *limiting process* in which the non-degenerated quadric is transformed in the degenerated quadric, which consists of the two circular points, the points at infinity which are incident with the isotropic lines Laguerre already examined. In this way Cayley defined the measure of angles by formulas which are equivalent to those of Laguerre. – At the end of his memoir Cayley wrote his now famous statement: "Metrical geometry is thus part of descriptive geometry [i.e. projective geometry] and descriptive geometry is all geometry and reciprocally." Let us remark that Cayley did not recognized the connection between his metrics and Lobatschewskian geometry and that the limiting process he used is somewhat dubious.

Klein took up the idea of Cayley and discovered that not only the Euclidean geometry but also the then known non-euclidean geometries, the Lobatschewskean geometry and the elliptic geometry can be founded in projective geometry. In his systematic studies he advanced in two steps: First he defines a metric in a projective space and secondly he characterised substructures in the previous defined projective-metric spaces. These substructures Klein named "Eigentlichkeitsbereiche". Today they are called *Cayley-Klein geometries*.

To define a projective metric Klein started with an arbitrary conic C in the complex projective plane. If C is real he got the hyperbolic geometry, if C is imaginary the elliptic geometry and if C is degenerated the euclidean geometry. As there are seven curves of second degree in the real projective plane there are also seven metrics. – To define "Eigentlichkeitsbereiche" Klein classified metrics in the following way: He called a metric on a (not absolute) line *l* *elliptic* or *parabolic* or *hyperbolic* if *l* is incident with no or one or two real absolute points. He called a metric in a (not absolute) point *P elliptic* or *parabolic* or *hyperbolic* if *P* is incident with no or one or two real absolute lines. Then he could define an „Eigentlichkeitsbereich" as a set T_0 of (non absolute) points and a set T_1 of (non absolute) lines of the projective plane such that

(i) the elements of T_0 can be mapped into each other by a motion (a collineation which maps the absolute into itself)

(ii) the elements of T_1 are incident with at least one point of T_0 and have a common metric (elliptic, parabolic or hyperbolic).

One can show that there are exactly nine plane Eigentlichkeitsbereiche („Cayley-Klein geometries“):

ml mp	elliptic	parabolic	hyperbolic
elliptic	*elliptic*	*euclidean*	*hyperbolic*
parabolic	*coeuclidean*	*galilean*	*cominkowskian*
hyperbolic	*cohyperbolic*	*minkowskian*	*double-hyperbolic*

“ml” means metric on a line and “mp” metric in a point. – In a 3-dimensional projective space there are 27 geometries, in a n-dimensional projectives space 3^n (Sommerville [1910/11]).

Let me finish this section with some remarks to the *relevance of the Cayley-Klein geometries*.

(1) Cayley and Klein created the first *systematic approach* to non-euclidean geometry – and not only special examples of such geometries.

(2) The Klein models convinced mathematicians and philosophers of the “*existence*” of non-euclidean geometry.

(3) The non-euclidean geometries have *equal status* as the euclidean geometry: they differ only in the choice of the absolute conic.

(4) The non-euclidean geometries are paradigmatic examples in the Erlangen Programme of F. Klein: The different models of one and the same geometry are examples for his transformation principle (“Übertragungsprinzip”) and show the characterizing relevance of transformations for a geometry.

(5) Klein´s group-theoretical approach to geometry can be seen as a principal antecedent of the modern formalistic conception of mathematics (according to Hilbert) in so far as the status of the objects of a geometry are irrelevant and only the relations are of importance.

2. A modern reconstruction of projective spaces with Cayley-Klein metrics

In the following a projective space $\mathcal{P}$ is always a pappian projective space of finite dimension, where the axiom of Fano holds. An axiomatic characterization of projective spaces as incidence structures of points and lines is given in Veblen and Young [1910] (see also Lenz [1965]).

An approach, where points and lines are not distinguished from subspaces of other dimension, is given in lattice theory. Following this approach (see e. g. G. Birkhoff [1973^3] or C.-A. Faure and A. Frölicher [2000]) we think of a projective space $\mathcal{P}$ as a complemented atomic modular lattice (V, *Ç* , *È*) of finite length. The elements of V are called flats or *subspaces* of $\mathcal{P}$. The subspace, which has only one element (the nullvector 0), and the whole space $\mathcal{P}$ are called *universal bounds* and are denoted by 0 and 1. α *Ç* β = γ means that γ is the subspace of maximal dimension of $\mathcal{P}$, which lies in α as well as in β. α *È* β = γ means that γ is the subspace of minimal dimension of $\mathcal{P}$, which has α and β as subspaces.

The *dimension* of a subspace is the projective dimension, which is one lower than the lattice-theoretic dimension. We call the subspaces of dimension 0, 1, 2, n-1 *points*, *lines, planes* and *hyperplanes* respectively. The partial ordering *Ì* of the lattice $\mathcal{P}$ is the incidence relation of the projective space. We write α *Í* β if α *Ì* β or α = β. If α, β are subspaces of V with α *Ì* β and, then the *interval* [α, β] = {γ *Î* V: α *Í* γ *Í* β} is again a projective space. - The projective space (V, *È* , *Ç*), the dual of $\mathcal{P}$, is denoted by $\mathcal{P}^*$.

In the lattice-theoretic approach some concepts of projective geometry need (minimal) extensions. We mention, that *collineations* are automorphisms of the lattice, which fix 0 and 1, while *correlations* are anti-automorphisms, which interchange 0 and 1. A projective correlation π of order 2 is called an *elliptic polarity*, if there are no self-conjugate points (if α is of dimension 0, then α *Ç* $\alpha\pi = 0$). A projective correlation π of order 2 is called a *hyperbolic polarity*, if there are points α, β with α *Ç* $\alpha\pi = 0$ and β *Ç* $\beta\pi \neq 0$.

From an analytic point of view, V has a (commutative) field K of coordinates of characteristic $\neq 2$ and can be represented as the lattice of subspaces of the vector space K^{n+1}. Elliptic and hyperbolic polarities can be described by symmetric bilinear forms.

We now give a synthetic definition of the general concept of a projective space with Cayley-Klein metric:

Definition: $(\mathcal{P}, (([\varepsilon_0, \varepsilon_1], \pi_1), ... ([\varepsilon_r, \varepsilon_{r+1}], \pi_{r+1})))$ with $r \geq 0$ is a *projective space of dimension n Î IN with Cayley-Klein metric*, if the following assumptions hold:

(i) $\mathcal{P}$ is a projective space of dimension n.

(ii) ε_0 , ε_1, ... ε_{r+1} are subspaces of $\mathcal{P}$ with $0 = \varepsilon_0 \subset \varepsilon_1 ... \subset \varepsilon_{r+1} = 1$.

π_k (with $1 \leq k \leq r+1$) is a hyperbolic or elliptic polarity on the interval $[\varepsilon_{k-1}, \varepsilon_k]$.

Examples: (a) a 3-dimensional projective space with an euclidean metric: $(\mathcal{P}, (([\varepsilon_0, \varepsilon_1], \pi_1), ([\varepsilon_1, \varepsilon_2], \pi_2)))$ with a 3-dimensional projective space P, a hyperplane ε_1 (of dimension 2) and elliptic polarities π_1, π_2.

(b) a 4-dimensional projective space with a Minkowskian metric: $(\mathcal{P}, (([\varepsilon_0, \varepsilon_1], \pi_1), ([\varepsilon_1, \varepsilon_2], \pi_2)))$ with a 4-dimensional projective space P, a hyperplane (of dimension 3), a hyperbolic polarity π_1 and an elliptic polarity π_2.

Now we can introduce the metric concepts of a polar and of a motion.

Definition: Let $(\mathcal{P}, (([\varepsilon_0, \varepsilon_1], \pi_1), ... ([\varepsilon_r, \varepsilon_{r+1}], \pi_{r+1})))$ be a projective space with Cayley-Klein metric and α a subspace. A subspace β is called a *polar of* α, if the following condition holds (for $1 \leq k \leq r+1$):

$(\beta \cap \varepsilon_k)$ *È* $\varepsilon_{k-1} = ((\alpha \cap \varepsilon_k)$ *È* $\varepsilon_{k-1})\, \pi_k$.

Example: $(\mathcal{P}, (([\varepsilon_0, \varepsilon_1], \pi_1), ([\varepsilon_1, \varepsilon_2], \pi_2)))$ a 3-dimensional projective space with an euclidean metric. Let α be a point not incident with ε_1 and β the plane ε_1. Then is β is a polar of α, because

k = 1: $(\beta \cap \varepsilon_1)\ \grave{E}\ \varepsilon_0 = \varepsilon_1\ \grave{E}\ 0 = \varepsilon_1 = \varepsilon_0\,\pi_1 = (\varepsilon_0\ \grave{E}\ 0)\,\pi_1 = ((\alpha \cap \varepsilon_1)\ \grave{E}\ \varepsilon_0)\,\pi_1$

k = 2: $(\beta \cap \varepsilon_2)\ \grave{E}\ \varepsilon_1 = \varepsilon_1\ \grave{E}\ \varepsilon_1 = \varepsilon_1 = \varepsilon_2\,\pi_2 = (\alpha\,\grave{E}\ \varepsilon_1)\,\pi_2 = ((\alpha \cap \varepsilon_2)\ \grave{E}\ \varepsilon_1)\,\pi_2$

Definition: Let β be a polar of α with $\alpha \cap \beta = 0$. Then the harmonic homology $\sigma_{\alpha,\beta}$ with α and β as center and axis is called a *reflection*. The group, generated by all reflections $\sigma_{\alpha,\beta}$ is called the *group of motions*.

In the case of three dimensions we get 18 spaces (for 3-dimensional spaces over the field of real numbers compare Yaglom et al. [1964], Giering [1982], and Rosenfeld [1988]):

$C(\varepsilon_0, .., \varepsilon_{r+1})$	**$C(\varepsilon_0, \varepsilon_1)$**	**$C(\varepsilon_1, ..., \varepsilon_{r+1})$**
elliptic	elliptic space	-
hyperbolic	hyperbolic space	-
hyperbolic of index 2	hyperbolic space of index 2	-
euclidean	elliptic plane	point
minkowskian[1)]	hyperbolic plane	point
quasielliptic	elliptic line	elliptic line
quasihyperbolic	elliptic line	hyperbolic line
galilean	elliptic line	euclidean line
quasielliptic of index 1	hyperbolic line	elliptic line
quasihyperbolic of index 1	hyperbolic line	hyperbolic line
galilean of index 1	hyperbolic line	euclidean line
coeuklidean[2)]	point	elliptic plane
cominkowskian[3)]	point	hyperbolic plane
isotropic	point	euclidean plane
isotropic of index 1	point	minkowskian plane
dual-galilean	point	coeuklidean plane
dual-pseudogalilean	point	cominkowskian plane
flag space	point	galilean plane

[1)] pseudoeuclidean, [2)] dual-euclidean, [3)] dualpseudoeuclidean

In a projective space with a Cayley-Klein metric one can define Eigentlichkeitsbereiche in a similar way as Klein did. Thus the concept of a non-euclidean geometry developed by Cayley and Kein can be reconstructed in a precise manner (cf. H. Struve and R. Struve [2004]).

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