In Search of Formalism

Lecture I: Background

École Normale Supérieure

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Michael Detlefsen
mdetlef1@nd.edu

University of Notre Dame
Agence Nationale de la Recherche
U of Paris-Diderot
U of Nancy 2
Collège de France

A Concern with Form ...

- Formalism is a concern with forms
  - Different types of forms
    - Most important for us are those Hilbert broadly referred to as "formal thought processes" (*formaler Denkprozesse*)
    - Processes that represent a "high degree of abstraction"
  - Different concerns about them
    - What are these "Denkprozesse"?
    - In what sense(s) do they represent a "high degree of abstraction" in our thinking?
    - What role(s) do forms of such "high degree of abstraction" play in mathematics?
    - By virtue of what?
    - How well?
    - Why do we want such roles to be played?
    - What similarities/differences are there to other uses of form or abstraction in mathematics?

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Formalization as Record of The Technique of Actual Thinking

"The formula game (*Formelspiel*) that Brouwer judges so dismissively (*so wegwerfend urteilt*) has, besides its mathematical value, an important general *philosophical significance*. For this formula game is carried out (*vollzieht*) according to certain definite rules, in which the technique of our thinking (*Technik unseres Denkens*) comes to expression (*zum Ausdruck kommt*). These rules form a closed system that can be discovered and definitively stated. The fundamental idea (*Grundidee*) of my proof theory is none other than to describe the activity of our understanding, to make a protocol (*Protokoll*) of the rules according to which our thinking actually proceeds (*unser Denken tatsächlich verfährt*). Thinking, it so happens, parallels speaking and writing through the formation and stringing together (*Bildung und Aneinanderreihung*) of statements (*Sätzen*).

Hilbert, "Die Grundlagen der Mathematik" (1928), 15 (published text of 1927 Hamburg lecture)

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Formulas & the Technique of Thinking

According to Hilbert, then, formalization gives an explicit record or representation of the *technique* according to which our thinking actually proceeds.

He also said, however, that formulas are used (and useful) for more than the explicit recording or representation of mathematical reasoning.

They, or, somewhat more accurately, their use partially *constitutes* or comprises our thinking.

He described the way in which the use of formulas constitutes a part of our thinking by saying that it was a kind of "living" application of the the axiomatic method.

In addition, he said, such use of formulae is (partially) constitutive of more than just mathematical thinking.

It comprises an important part of everyday thinking as well, albeit a largely unconscious one.

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Formal Reasoning as Constitutive of the Technique of our Thinking

"In our theoretical sciences we are accustomed to the use of formal thought processes (*formaler Denkprozesse*) and abstract methods . . . [but] already in everyday life (*täglichen Leben*) one uses methods and concept-formations (*Begriffsbildungen*) which require a high degree of abstraction and which are only comprehensible (*Verständlich*) through unconscious (*unbewußte*) application of the axiomatic method. Examples include the general process of negation and, especially, the concept of infinity.

Hilbert, "Naturerkennen und Logik" (1930), 380 (page reference to reprinting in *Gesammelte Abhandlungen* III)

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Formalization & Formal Reasoning

According to Hilbert, then, there are two types of formulas in mathematics.

I. Those used in *formale Denkprozesse*, which processes partially *constitute* our thinking (and form its *technique*).

II. Those by means of which we *represent* our thinking as a system governed by explicitly given rules for manipulating symbolic expressions.

Comment:

- *Formale Denkprozesse* may thus be taken to include the explicit formalizations of mathematical thinking offered by Whitehead and Russell and others.

  [“In our theoretical sciences we are accustomed to the use of formal thought processes (*formaler Denkprozesse*) . . .]

- But they also be taken to include that *inexplicit reasoning* which, in Hilbert’s phrase, is *“an unconscious (unbewußte) application of axiomatic method”*.

  [“ . . . already in everyday life one uses methods and concept-formations (*Begriffsbildungen*) which . . . which only become intelligible by means of an unconscious (*unbewußte*) application of axiomatic methods.”]

The Technique of our Thinking

One of tne important roles of (the use of) formulas is as (partially) constitutive of the *technique* of our thinking.

But what *is* a *technique* of thinking?

Dictionaries characterize *technique* as . . .

"the mechanical or formal part of an art"

"manner of execution or performance in relation to formal or practical details"

"a characteristic way of proceeding"

They also tell us that technique is in some respects a superficial part of thought.

Example:

"A player may be perfect in technique, and yet have neither soul nor intelligence."

Grove, *A Dictionary of Music and Musicians* (1878–1889) IV, 66 (as referred to in the *OED*)

Soul and Intelligence . . . ?

If technique in thinking is like technique in music, then, it bespeaks neither soul nor intelligence on the part of the artist.

Hilbert said things that indicated a degree of agreement with this.

"In our science, it is always and only the reflecting mind (*der überlegende Geist*), not the applied force of the formula, that is the condition of a successful result."

Hilbert, letter to Minkowski

So too did others.

The following remark of Dedekind's is typical.

"A theory based on calculation would, so it seems to me, not offer the highest degree of perfection; it is preferable . . . to seek to draw the demonstrations, no longer from calculations, but directly from the characteristic fundamental concepts , and to construct the theory in such a way that it will, on the contrary, be in a position to predict the results of the calculation . . . ."

Dedekind, "Sur la theorie des Nombres entiers algebrique" (1877)

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The Importance of Technique I

These last remarks suggest a view that sees the use of formulas as being in important respects inferior to thinking which proceeds "directly from the characteristic fundamental concepts" of the subject to which it pertains.

Perhaps, then, the ubiquity of technique claimed by Hilbert ought to be regarded as a misfortune and something to be remedied.

8 $\longrightarrow$

The Importance of Technique II

On balance, though, this does not seem to have been how either Hilbert or Dedekind saw things.

In the passage from the Hamburg lecture quoted earlier, Hilbert said that the "formula game" that Brouwer peremptorily dismissed had "mathematical value".

Nor was this an uncharacteristic statement. In the passage quoted above from "Naturekennen und Logik", Hilbert characterized *formaler Denkprozesse* as representing a high degree of abstraction. He also praised the use of such abstraction as necessary to the attainment of the higher forms of science.

9 $\longrightarrow$

The Importance of Technique III

Hilbert also said that the use of methods of a high degree of abstraction—the salient feature of symbolic reasoning identified in his statement from "Naturerkennen und Logik"—was necessary for the full grasp and mastery of arithmetical concepts and proof methods.

"Arithmetical concepts and proof methods require (*erfordern*) for their grasp (*Auffassung*) and full mastery (*völligen Beherrschung*) a high degree of abstractive capacity (*Abstractionsfähigkeit*) of the mind (*Verstandes*), and this circumstance has occasionally been used as a reproach against arithmetic. I am of the view that all the other branches (*Wissensgebiete*) of mathematics require a similarly high degree of the mind's abstractive capacity—assuming that one also in these areas approaches the investigation of foundations with the strictness (*Strenge*) and completeness (*Vollständigkeit*) that is actually necessary."

"Die Theorie der algebraischen Zahlenkörper (1894–95)

In addition to this, of course, there was that greatest of all "creations of genius" in mathematics—the method of ideal elements—which consisted essentially in the use of formal reasoning.

The Importance of Technique IV

The supreme example of the method of ideal elements was in logic, where we added formal devices in order to facilitate use of those patterns of logical reasoning according to which our minds most efficiently, and most productively, operate.

"... we have an urgent reason for ... extending the formal point of view of algebra to all of mathematics. For it is the means of relieving us of a fundamental difficulty that already makes itself felt in elementary number theory. ...

... if we adopted the finitist attitude, we could not make use of the alternative according to which an equation ... in which an unspecified numeral occurs either is satisfied for every numeral or can be refuted by a counterexample. ... as an application of the "principle of excluded middle", this alternative depends essentially on the assumption that it is possible to negate the assertion that the equation in question always holds.

But we cannot relinquish the use either of the principle of excluded middle or of any other law of Aristotelian logic expressed in our axioms, since the construction of analysis is impossible without them."

"Grundlagen der Mathematik" (1927)

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Formal Thinking

If, then, as Hilbert said, it is only the "reflecting mind" and not the "applied force of the formula" that is the "condition of a successful result", he must have thought there was some way of linking formal processes of thinking to judgments of the "reflecting mind".

I'll leave consideration of this possibility for next time. Right now I want to make a few observations intended to focus attention on what I think is the salient finding thus far—namely, that those positions I am including under the heading of Formalism all share the following view in common.

Formal processes of thinking are not only useful and admissible for the representation of our thinking, but are also partially constitutive of our thinking.

In what exact sense this is so is, we have seen, more difficult to say.

Hilbert characterized it as a *technique* of thinking. By this he seems to have meant that it is constitutive of part of our thinking, but that its conclusions do not themselves constitute even part of the knowledge that it (formal thinking) ultimately provides for.

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A Second View of the Importance of Formal Reasoning

Heinrich Heine . . .

"Suppose I do not want to restrict myself to the positive rational numbers. I do not answer the question 'What is a number?' by defining number conceptually, say, by introducing irrationals as limits whose existence is to be presupposed. I adhere instead to the definition of the purely formal standpoint (*rein formalen Standpunkt*), in which what I call numbers are certain tangible (*greifbare*) signs (*Zeichen*) so that the existence of these numbers does not, therefore, stand in question. [Of course, MD] . . . the calculary operations (*Rechenoperationen*), and the number-signs (*Zahlzeichen*) must be so chosen, or equipped with such an apparatus as provides a grasp of the definition of these operations."

"Die Elemente der Funktionenlehre", *Journal für die reine und angewandte Mathematik* 74 (1872), 173

- The evidence which sustains judgment concerning certain non-semantical features of signs is of especially high quality.

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A Third View of the Importance of Formal Reasoning

Bernays . . .

"The inquiry into the foundations of mathematics has shown . . . that a certain kind of purely perceptual (*rein-anschaulich*) cognition must be taken as the starting-point for mathematics and that indeed one cannot develop even logic as the theory of judgements and inferences without resorting to some extent to such perceptual cognition. What is meant here is the perceptual representation of discreta (*des Diskreten*), an elementary form of perceptual cognition from which, in particular, we take our most primitive combinatorial representations."

"Die Grundgedanken der Fries'schen Schule in ihrem Verhältniß zum heutigen Stande der Wissenschaft", in *Abhandlungen der Fries'schen Schule*, neue folge, 1930, vol. 5

• • The evidence which sustains judgment concerning certain non-semantical features of signs is of especially high quality.

• • This evidence is capable of supporting a substantial body of mathematical belief and reasoning.

Efficiency & Reliability of Conductive Use of Formulae I

Leibniz . . .

". . . I hold [expressions concerning infinites, MD] to be mental fictions (*pro mentis fictionabus*), suited for use in calculations, like the imaginary roots in algebra. And yet I have demonstrated that these expressions (*expressiones*) are a great aid in shortening thought (*ad compendium cogitandi*) and also in discovery (*ad inventionem*), and it is not possible that they should lead us into error (*in errorem ducere non posse*)."

Letter to Fr. Des Bosses, March 17, 1707 (reprinted in Latin original in J. E. Erdmann (ed.), *Gottfried Wilhelm Leibniz: Opera Philosophica* (1840), 436

- There are non-semantical uses of signs in mathematics.
- These can have utility as calculating devices.
- In particular, they can increase epistemic efficiency . . .
- . . . and extent.
- And they can do so without sacrificing justificative quality.

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Instrumental Formalism: Sample Statement

Lambert ...

"No one has yet formed himself a clear representation of all the members of an infinite series, and no one is going to do so in the future. But we are able to do arithmetic with such series, to give their sum, and so on, by virtue of the laws of symbolic knowledge. We thus extend ourselves far beyond the borders of our actual (*wirklichen*) thinking. The sign $\sqrt{-1}$ represents an unthinkable non-thing. And yet it can be used very well in finding theorems. What are usually regarded as specimens of the pure understanding can be viewed most of the time as specimens of symbolic knowledge."

Letter to Kant, October 13, 1770

- There are non-semantical uses of signs in mathematics.
- These are in fact prevalent.
- These uses can have utility as calculating devices.
- In particular, they can increase epistemic extent.
- ... and also efficiency?
- ... and they can do so without sacrificing justificative quality?

Combined Views: Hilbert

There are also varieties of formalism that combine the Instrumental and Justificative variants.

Hilbert's foundational viewpoint is an example of this.

"In diametrical opposition to Frege and Dedekind, I find the objects of the theory of numbers in the signs themselves, whose form we can recognize universally and certainly, independently of place and time and of the special conditions attending the production of the signs as well as of insignificant differences in their elaboration. Here lies the firm philosophical orientation which I require as requisite to the grounding of pure mathematics, and to all scientific thinking, understanding, and communication. 'In the beginning,' we may say here 'was the sign'."

"Neubegründung der Mathematik" (1922), 162–163 (page references to reprint in *Gesammelte Abhand.*, vol. III)

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Formalism: Bloodline & Cohort

To understand what formalism is as an historical point of view in the foundations of mathematics, it is necessary to consider the ideas and viewpoints with which it has been associated.

1. Frege saw formalism as the most influential foundational view of his day. To understand its appeal, he believed, one had above all to appreciate the uncritical bias towards empiricism in his time. Anticipating the cool reception of his own foundational ideas he thus wrote:

   "... unpropitious for my book is the widespread inclination to acknowledge as existing only what can be perceived by the senses. That which cannot, people try to deny or else to ignore. Now the objects of arithmetic, i.e., numbers, cannot be perceived by the senses. How do we come to terms with them? Simplicity itself! We pronounce the numerical signs themselves to be the numbers. Then in the signs we have something visible, and that is naturally the chief thing."

   Grundgesetze (1893), XIII

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Frege on Formalism

"To quote Mill: "The doctrine that we can discover facts, detect the hidden processes of nature, by an artful manipulation of language, is so contrary to common sense, that a person must have made some advances in philosophy to believe it."

Very true—if we suppose that during the artful manipulation we do not think at all. Mill is here criticizing a kind of formalism that scarcely anyone would wish to defend. Everyone who uses words or mathematical symbols makes the claim that they mean something, and no one will expect any sense to emerge from empty symbols."

Frege, *Grundlagen* (1884), §§15-16, 22

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Thomae: Mathematics as Symbolic Game

"For the formalist, arithmetic is a game with signs, which are called empty. That means they have no other content (in the calculating game) than they are assigned by their behavior with respect to certain rules of combination (rules of the game). The chess player makes similar use of his pieces; he assigns them certain properties determining their behavior in the game, and the pieces are only the external signs of this behavior. To be sure, there is an important difference between arithmetic and chess. The rules of chess are arbitrary while the rules of arithmetic are such that by means of simple axioms numbers can be referred to perceptual manifolds and thus make an important contribution to our knowledge of nature."

Thomae, *Elementare Theorie der analytischen Functionen einer complexen Veränderlichen*, 2nd ed.(1898), 3

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Goethe on Signs & Failure Concepts

"Thus even where concepts fail, there appears a word just in the nick of time."

Goethe, *Faust* I (Mephistopheles)

["Denn eben wo Begriffe fehlen, Da stellt ein Wort zur rechten Zeit sich ein."]

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Bernays: The Methodological Principle of Hilbert's Program

"Where concepts fail, a sign is introduced in the nick of time. This is the methodological principle of Hilbert's theory. An example should explain what is meant. The existence axiom "for each number, there is a successor" holds in number theory. In accordance with the restriction to the concrete-intuitive, it is now a matter of avoiding the general concept of number as well as the existential form of the statement."

Bernays, 'Über Hilbert's Gedanken zur Grundlegung der Arithmetik', *Jahresberichte der deutsche Mathematiker-Verienigung* 30 (1922): 16

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