In Search of Formalism

Lecture II: Formal Thinking, Abstraction & The Axiomatic Method

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October 17, 2008

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I. Review

Two Uses of Formulae

Representative Use: Use of formal objects† ($\approx$ formulae) to *represent* reasoning (in order to facilitate its study) $\not\Longrightarrow$ use of formulae partially constitute reasoning

Conductive Use: Use of formulae to *conduct* reasoning $\Longrightarrow$ use of formulae partially *constitutes* reasoning

Claim: Formalism in the philosophy of mathematics $\Longrightarrow$ conductive use of formulae

Representative use of formulae, on the other hand $\not\Longrightarrow$ conductive use

$\therefore$

Representative use of formulae $\not\Longrightarrow$ formalism

However, the most influential version of formalism (i.e., Hilbert's) allowed an important role not only for conductive uses of formulae, but also representative. It was key to his proposed proof-theoretic justification of traditional conductive uses of formulae in mathematics.

Other versions (e.g. Heine's and Thomae's) did not include a significant representative role.

†: 'formulae' will abbreviate 'formal objects' ... whatever those are

II. Representative & Conductive Uses of Formulae

Representative Use of Formulae: Text

"The formula game (*Formelspiel*) that Brouwer judges so dismissively (*so wegwerfend urteilt*) has, besides its mathematical value, an important general *philosophical significance*. For this formula game is carried out (*vollzieht*) according to certain definite rules, in which the technique of our thinking (*Technik unseres Denkens*) comes to expression (*zum Ausdruck kommt*). These rules form a closed system that can be discovered and definitively stated. The fundamental idea (*Grundidee*) of my proof theory is none other than to describe the activity of our understanding, to make a protocol (*Protokoll*) of the rules according to which our thinking actually proceeds (*unser Denken tatsächlich verfährt*). Thinking, it so happens, parallels speaking and writing through the formation and stringing together (*Bildung und Aneinanderreihung*) of statements (*Sätzen*).

Hilbert, "Die Grundlagen der Mathematik" (1928), 15 (published text of 1927 Hamburg lecture)

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Conductive Use of Formulae: Texts I

In this same passage, Hilbert characterizes the formula games, or the rules that comprise them, as being an expression or manifestation of "the technique of our thinking".

"[Formula games are] carried out (*vollzieht*) according to certain definite rules, in which the technique of our thinking (*Technik unseres Denkens*) comes to expression (*zum Ausdruck kommt*)."

Technique, we saw last time, is commonly understood as "the mechanical or formal part of an art" or the "manner of execution or performance in relation to formal or practical details".

Applying this common understanding here, we understand *the technique of our thinking* to signify a mechanical or formal part of our thinking.

On this understanding, to say that formula games are the "technique of our thinking" made manifest is thus to say that they are a *part of our thinking* made manifest.

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Conductive Use of Formulae: Texts II

This conductive reading fits well with Hilbert's further remark concerning the fundamental idea of his proof theory.

"The fundamental idea (*Grundidee*) of my proof theory is none other than to describe the activity of our understanding, to make a protocol (*Protokoll*) of the rules according to which our thinking actually proceeds (*unser Denken tatsächlich verfährt*)."

The term 'protocol' generally means:

- an official *record* of events or procedures
- a procedural *norm* (e.g. a diplomatic etiquette)

Read either way, the formalizations with which Hilbert's proof theory was to deal were intended to identify and make explicit something he took to be *part of our mathematical thinking*.

Read the second way, there is the additional suggestion that what is already in our thinking is in some sense proper, or at least normal.

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Conductive Use of Formulae: Texts III

A general statement of the conductive use of formulae . . . and its desirability.

"[T]he deduction of the 'true' formulas from the axioms must be purely formal, in accordance with the axiomatic point of view, so that we are not at all concerned with the meaning of the sentences symbolized by the formulas, but solely with following the prescriptions contained in the rules. Only in the interpretation of the results obtained by the formal operations should we take into account the meaning of the symbols of our calculus."

Hilbert & Ackermann, *Grundzüge der theoretischen Logik* (1928), 67–68

Deductive reasoning itself is supposed to be . . .

- purely formal
- in accordance with the axiomatic point of view ♯
- not concerned with the meanings of the sentences symbolized by the formulae
- solely concerned with following the prescribed formal rules

Concern with meaning enters only when

- interpreting the *result* of the reasoning

Conductive Use of Formulae: Texts IV

Second general statement of the conductive use of formulae, in mathematics, and in thinking generally.

"In our theoretical sciences we are accustomed to the use of formal thought processes (*formaler Denkprozesse*) and abstract methods . . . [but] already in everyday life (*täglichen Leben*) one uses methods and concept-formations (*Begriffsbildungen*) which require a high degree of abstraction and which only become perspicuous (*Verständlich*) through unconscious (*unbewußte*) application of the axiomatic method. Examples include the general process of negation and, especially, the concept of infinity.

"Naturerkennen und Logik" (1930), 380 (page reference to reprinting in *Gesammelte Abhandlungen* III)

Hilbert thus suggested a similarity between . . .

- use of formal thinking processes

  and

- use of abstract methods

And in saying that we manage abstractions by applying the axiomatic method, he suggested a related parallel between

- • *formal* thinking and *axiomatic* thinking.

Conductive Use of Formulae: Texts V

"If we now . . . set our sights on elementary number theory, we recognize that we can obtain and prove its truths through contentual intuitive (*inhaltlich–anschauliche*) considerations. The formulas that encounter when we take this approach are used only to impart information. Letters stand for numerals and an equation informs us of the fact that two signs stand for the same thing.

The situation is different in algebra . . . Where we had numerals, we now have formulas, which themselves are concrete objects that in their turn are considered by our perceptual intuition, and the derivation (*Ableitung*) of one formula from another in accordance with certain rules (*nach gewissen Regeln*) takes the place of contentual number-theoretic proof. . . .

Hence even elementary mathematics contains . . . formulas that . . . in themselves mean nothing but are merely things governed by our rules."

"Die Grundlagen der Mathematik" (1928), 7–8

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III. Formulae, their Conductive Use & Axiomatic Thinking

Contentual Thinking & Formal Thinking I

These texts give evidence that Hilbert believed in the conductive use of formulae.

But what did he take this use to be?

And, for that matter, what did he take formulae to be?

Part of answer provided by contrast Hilbert made between what we are calling the *conductive* use of formulae and what he described as their *contentual* use.

Here the term ‘conductive’ might cause confusion, so caution is in order.

Both what I’m calling the conductive use of formulae and what Hilbert termed their contentual use are in a broader sense conductive uses—that is, they are both conceptions of the role(s) formulae play in the conduct of reasoning.

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Contentual Thinking & Formal Thinking II

Difference: In the contentual conception, it's not formulae but their contents that are doing the work.

What the formulae contribute to reasoning is their contents. Reasoning or inference itself is a relation between these contents.

In the conductive conception, formulae contribute something else—though exactly what else we have yet to say, and it is difficult to say.

As a first approximation, we might say that they contribute their forms.

But saying this doesn't take us very far without a clearer characterization of what this 'form' is ... and a clearer characterization of what a formula is.

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Contentual Use of Formulae & Formal Use of Formulae

The basic distinction.

- contentual use of formulae in reasoning ≈ use in which
  - formulae are treated as having contents

  and

  - a step of reasoning (i.e. an inference) from one formula to another is warranted only if the content of the one is justifiedly judged to logically imply the content of the other.

- formal use of formulae ≈ use in which
  - formulae are treated only in terms of their non-contentual properties according to established rules formulated in terms of those properties

  and

  - addition of a formula to a sequence at a given place is warranted only if there is justified judgment of what the non-contentual properties of the involved formulae are and justified judgment that the addition results from a correct application of the rules.

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Terminology

What I have been calling the *conductive* use of formulae is thus, strictly speaking, the *formal conductive* use of formulae.

This noted, I'll continue referring to *formal conductive* use of formulae as simply *conductive* use.

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Formal Thinking and Axiomatic Thinking I

Let's consider this more closely.

Conductive use of formulae $\approx$ use which treats formulae only in terms of their non-contentual properties according to established rules formulated in terms of those properties.

But is this characterization unnecessarily—and perhaps undesirably—weak?

Why not a characterization that says what conductive use *is*, rather than only what it *is not* (i.e. not contentual processing) . . . a characterization such as this:

Conductive use of formulae $\approx$ use of formulae which treats them only in terms of their *syntactical properties* according to established *rules formulated in syntactical terms*.

There are both logical and textual reasons for not doing so.

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Formal Thinking and Axiomatic Thinking II

Logically, there is no evident reason why non-contentual forms, the forms processed by *formaler Denkprozesse*, should be restricted to syntactical forms.

Textually, there are repeated indications that Hilbert did not restrict *formaler Denkprozesse* to processes dealing with syntactical forms.

We've already seen one example of this.

"...already in everyday life (*täglichen Leben*) one uses methods and concept-constructions (*Begriffsbildungen*) which require a high degree of abstraction and which are only comprehensible (*Verständlich*) through unconscious (*unbewußte*) application of the axiomatic method ..."

Here Hilbert characterizes a formal thinking process as an unconscious application of the axiomatic method.

For him, though, application of the axiomatic method and processing in terms of syntactical characteristics were different things.

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Formal Thinking and Axiomatic Thinking III

"…already in everyday life (*täglichen Leben*) one uses methods and concept-constructions (*Begriffsbildungen*) which require a high degree of abstraction and which are only comprehensible (*Verständlich*) through unconscious (*unbewußte*) application of the axiomatic method …"

Hilbert thus counted as *formaler Denkprozesse*, such things as the application of the axiomatic method to concept-constructions, as distinct from syntactic objects.†

The particular form that *formaler Denkprozesse* took in these cases was that of an application of the axiomatic method.

Hilbert thus allowed for forms other than syntactic forms to be the forms dealt with in *formaler Denkprozesse*.

†: More on his conception of axiomatic method later.

Formal Thinking and Axiomatic Thinking IV

He also stressed the importance of non-syntactic forms in his correspondence with Frege.

"... it is surely obvious that every theory is only a scaffolding or schema of *concepts* together with their necessary relations to one another, and that the basic elements can be thought of in any way one likes. If in speaking of my points I think of some system of things, e.g. the system: love, law, chimney sweep ... and then assume all my axioms as relations between these things, then my propositions, e.g. Pythagoras' theorem, are also valid for these things."

Letter to Frege, Dec 29, 1899

Hilbert thus saw theories as themselves being forms—forms constituted by *abstract relations between concepts*.

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Formal Thinking and Axiomatic Thinking IV

Bernays put the "theories are forms" point this way:

"A main feature of ... [Hilbert's understanding of] ... the axiomatic method is presented and practiced in the spirit of the abstract conception of mathematics that arose at the end of the nineteenth century and which has been generally adopted in modern mathematics. It consists in abstracting from the meanings of the terms ... and ... the fundamental relations and ... understanding the assertions (theorems) of the axiomatized theory in a hypothetical sense, that is, as holding true for any interpretation or determination of the kinds of individuals and of the fundamental relations for which the axioms are satisfied. ... an axiom system is regarded not as a system of statements about a subject matter but as a system of conditions for what might be called a relational structure. Such a relational structure is taken as the immediate object of the axiomatic theory ..."

"Hilbert, David", *Encyclopedia of Philosophy*, vol. III, 497

Bernays cited Hilbert's axiomatization of geometry as paradigmatic example of this conception of axiomatic thinking.

Formal Thinking and Axiomatic Thinking: Summary

Bernays' description of Hilbert's conception of axiomatic method is a good summary statement of it.

- It exemplifies the modern formal or abstract spirit in mathematics
- It represents a type of formal reasoning (*formaler Denkprozesse*)
- The forms in terms of which it proceeds are not syntactical forms but abstract relations between concepts

  so

- Theories taking this form are not formal systems in the contemporary sense, but only informal axiomatic theories (like Hilbert's axiomatization of geometry)
- Conductive use of forms in reasoning $\not\Longrightarrow$ conductive use of formulae in reasoning
- Reasoning in the modern abstract spirit $\not\Longrightarrow$ conductive use of formulae

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Formalism & Formal Thinking: Conclusions & Questions

Let's now consider this finding:

1. Conductive use of forms in reasoning $\not\Longrightarrow$ conductive use of formulae in reasoning

I claimed last time that

2. Formalism $\Longrightarrow$ conductive use of formulae

1. & 2. together imply that

Conductive use of forms in reasoning $\not\Longrightarrow$ formalism

Formalism requires more than conductive use of the types of forms representative of the modern axiomatic method.

What more?

And can this extra element be justified?

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The Additional Element(s)

Part of the answer to the question "What more does formalism require than merely conductive use of forms?" is probably not surprising by now.

It is: Conductive use of formulae . . . where formulae here are now to be understood as formal-syntactical items.

The other part of the answer may be more surprising.

It is: Representative use of formulae.

What I am calling formalism thus requires both conductive use of formulae and representative use of formulae.

Formalism $\Longrightarrow$ conductive use of formulae + representative use of formulae

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Clarification I

Formalism $\Longrightarrow$ conductive use of formulae + representative use of formulae

This may be misleading.

It might be thought that the respect or sense in which formalism requires conductive use of formulae is different from the respect or sense in which it requires the representative use.

Conductive use of formulae is logically a part of what formalism is.

The implication in

Formalism $\Longrightarrow$ conductive use of formulae

is thus broadly logical implication.

Is the implication in

Formalism $\Longrightarrow$ representative use of formulae

also of this sort?

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Clarification II

Here we squarely confront the need for a clearer statement of formalism.

As a first approximation, I offer the following:

There are conductive uses of formulae in mathematical reasoning that are both useful and reliable.

When formalism is stated this way, the implication

$$\text{formalism} \Longrightarrow \text{representative use of formulae}$$

is not broadly logical implication.

What is closer to the truth is that it is the justification of formalism, not formalism itself, that requires the representative use of formulae.

Even here, though, the nature of the requirement is not logical but pragmatic-epistemic in nature . . .

Roughly, the idea is that the only way we know to properly *justify* formalism involves representative use of formulae.

I'll use the following notation to express this idea:

$$\text{formalism} \rightsquigarrow \text{representative use of formulae}$$

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The Justification of Formalism

To justify formalism requires that we show the existence of conductive use of formulae that are useful and reliable.

What does it mean to say that a conductive use of formulae is *useful*?

Hilbert's general answer was 'Abkurzung' or 'Denkökonomie' (cf. e.g. "Naturerkennen und Logik", 380).

He didn't say in greater detail in what he took such "abbreviation" or "economy of thought" to consist.

He did, though, make some general statements about it.

And he also described some examples.

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IV. The Nature & Value of Conductive Use of Formulae

Functions of Conductive Use of Formulae

- Means or medium of reasoning (Schlußweisen) (1926, 162 [370])
- Guide (Anweisung) to our thinking (1926, 167 [373])
- Practical instrument (praktisches Instrument) of thinking (1926, 167 [373])
- An 'apparatus' (ein Apparat) for the conduct of thinking (1926, 171 [376–77])

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Nature & Origin of Conductive Use

- Created . . . 'children' of the human mind (Kind des menschlichen Geistes) (1926, 165 [372])
- Arise from concepts (Begriffe) in our thinking (in unserem Denken) (1926, 165 [372]). Recall Kant.

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Value of Conductive Use: General

- Indispensable (unentbehrliches) instruments of human thinking (1926, 165 [372]; 174 [379])
- Well-justified (wohlberechtigten) instruments of human thinking (1926, 165 [372])
- Unerring guides (unfehlbare Anweisung) to human thinking (1926, 167 [373])

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Value of Conductive Use: Specific

- Maximize simplicity of (∴ ability to apply and remember?) methods of thinking (so einfach wie nur irgend möglich machen) (1926, 166 [372–73])
- Maximize perspicuity or surveyability of (∴ ability to apply and remember?) methods of thinking (so übersichtlich wie nur irgend möglich) (1926, 166 [372–73])
- Maximize familiarity or permanence of (∴ ability to apply and remember?) methods of thinking (1926, 166 [372–73]; 174 [379])

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V. Examples of Valuable Conductive Use

Example: Elementary Plane Geometry

Real Objects: points, lines

Law of Connection: For any two points, there is one and only one line that is their join.

Principle of Duality (PD): Every theorem (proof) of elementary plane geometry gives way to another theorem (proof) by substituting 'line' for 'point', 'point' for 'line', 'meet' for 'join' and 'join' for 'meet'.

Conductive Use: Add term for point 'at infinity' to serve as the meet of parallel lines and, so, preserve PD.

Result: Preserve PD. Hence, for example, obtain the *Law of Intersection* (for any two lines, there is one and only one point that is their meet) from the *Law of Connection* by simple scheme of substitutions. Generally, obtain two proofs for the investigative costs of one.

Gain: Efficiencies of proof?

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Example: Complex Numbers

The most appealingly simple law concerning the existence and number of roots of equations is the **Fundamental Theorem of Algebra** (FTA): *Every equation of degree n has exactly n roots.*

Conductive Use: Introduce terms for imaginary quantity $i = \sqrt{-1}$ and its complex cohort.

Result: Preserve FTA for polynomial equations generally (e.g. for equations such as $x^2 + 1 = 0$).

Gain: Simplicity of law(s) governing roots of equations? Resultant efficiencies of problem-solving (= root finding)?

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Example: Classical Logic

The non-classical logical relations that hold among *real propositions* are, generally speaking, imperspicuous.

The most perspicuous (most familiar and simplest) logical laws are the laws of classical logic.

Conductive Use: Add the so-called **ideal propositions**—e.g. unbounded existential quantifications—to **real propositions** and apply the laws of classical logic (e.g. the law of excluded middle) to the resulting scheme of 'propositions'.

Result: Preservation of the simple, perspicuous, familiar laws of classical logic as the laws of mathematical reasoning.

Gain: General efficiencies of mathematical thinking? Discovermental efficiencies particularly increased?

Example: Genetics & Geometry

One of Hilbert's favorite examples of the utility of formal thinking was the applicability of the linear congruence axioms of Euclidean geometry† to the laws determining trait-coupling in deviant Drosophila offspring.

Hilbert described this convergence of form as "more wonderful" than anything "imagined in the boldest fantasy" ("Naturerkennen und Logik", 380).

†: E.g. every line segment is congruent to itself, two segments congruent to a third are congruent to each other

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Observations & Questions

I. The above are all examples Hilbert gave of the economies of thought offered by the use of ideal methods. All of them, however, seem to deal with economies of thought that are duplicated in informal theories that are present not only in formal theories, but in informal theories parallel to them.

Do they then constitute examples of the economizing effects of conductive uses of formulae?

Or must it be shown that there are economies of thinking that are due exclusively to the use of formulae and not duplicated in parallel informal reasoning?

II. Are the economies of thinking represented by the above examples all, at bottom, the same type of economy? Or are there differences between them that are just as significant as the similarities? Are there other types of *Denkökonomie* that matter, or ought to matter, as much or more?

III. It seems that economy of thinking is valuable only when the means of attaining it don't sacrifice epistemic quality. What types of quality figure here, and what types of conditions might be placed on conductive uses of formulae in order to secure these?