In Search of Formalism

Lecture III: Forms & Their Value: Economy of Thought

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I. Review/Preview

Review/Preview I

We thought about forms and formulae, and noted that

- There is significant conductive use of types of forms (specifically, informal abstract axiomatic forms) other than formulae in mathematical reasoning.
- This is in the spirit of modern mathematics.

$\therefore$

- Some conductive use of forms in the spirit of modern mathematics $\not\Longrightarrow$ conductive use of formulae.
- Formalism $\Longrightarrow$ conductive use of formulae

[“The essence of formalism is that we *use* symbols according to definitely prescribed rules, not that we talk about symbols.”
H. B. Curry, “Mathematics, Syntactics and Logic”, *Mind* 62 (1953), 175 (emphasis Curry’s)]

$\therefore$

- Some conductive use of forms in the modern abstract spirit $\not\Longrightarrow$ formalism

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Review/Preview II

This left us with two questions.

Descriptive Question: Does modern mathematics make significant conductive use of formulae?

Answered "yes" last time, and surveyed some examples from Hilbert. Today we'll take a closer look.

Prescriptive Question: Ought it to do so? That is, would there be compelling value in its doing so?

We'll focus on the Prescriptive Question, though, and try to get clearer on what the *value* of conductive use of formulae might be.

Remark: Of special interest will be a suggestion that the value of the *use of forms* generally depends on its being essentially a *use of formulae*.

Review/Preview III

We also pointed out both a difference and a dependency in the roles that representative and conductive uses of formulae play in formalism. Specifically . . .

Formalism requires both conductive and representative use of formulae.

But it requires them in different ways.

It logically requires conductive use.

formalism $\Longrightarrow$ conductive use of formulae

But it only justificatively requires representative use; that is

the justification of formalism $\rightsquigarrow$ ($\approx$ practically requires) representative use of formulae.

This led us to the following preliminary formulation of formalism.

There is extensive conductive use of formulae in mathematical reasoning that is both

(i) valuable as a means of conserving resources, and

(ii) reliable as a means of determining what to believe.

Preview/Review IV

Stated this way, the defense of formalism can be divided into three subtasks.

i. Show that conductive use of formulae is extensive
ii. Show that it is valuable
iii. Show that it is reliable

Our focus today will be the second subtask—in particular, the question of what the value of the use of formulae in mathematical reasoning consists in.

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II. The Nature & Value of Conductive Use

The Nature & Value of Conductive Use I

Let's briefly recall some of the things Hilbert said about the nature and value of conductive use of formulae.

Nature

Practical instrument of conclusory† thinking

- *Schlußweisen* (1926)†, 162
- *praktisches Instrument* (1926), 165, 167
- *ein Apparat* (1926), 171

†: Conclusory thinking ≈ thinking relating or tending to a conclusion

†: "Über das Unendliche", 1926 version (*Mathematische Annalen* 95). Shortened version published in 1927 (*Jahresbericht* 36).

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The Nature & Value of Conductive Use II

Conductive use of formulae is a *practical instrument* of conclusory thinking.

What makes it valuable?

Mainly, that it's *guided* (*Anweisung* (1926), 167) . . .

. . . and guided well.

Specifically, it's . . .

- Guided efficiently (*unentbehrliches* (1926), 165, 174)

  Helps us get the greatest conclusory product per unit resource consumed

  Not to conductively use formulae might even seriously impair development of knowledge. As Hilbert put it, "it is an *indispensable* (*unentbehrliches*, (1926), 165, 174)" instrument of thought.

- Guided reliably, and certifiably so

  Conductive use of formulae can be trustworthy (indeed, unerring, *unfehlbare* (1926), 167) as regards contentual conclusions. And we can know this.

The Nature & Value of Conductive Use: Summary

In sum, conductive use of formulae can have a well-justified (*wohlberechtigten* (1926), 165) place in our thinking.

Specifically, the thinking goes, it can be justified by appeal to its

- efficiency

and its

- reliability

We'll now consider each of these more carefully, and also note an important connection between them—namely, that

- it is the efficient conductive uses of formulae the formalist wants to preserve.

Therefore, at least in principle,

- she has no intrinsic programmatic interest in defending the reliability of inefficient uses.

Core Task Regarding Reliability: Establish it for those conductive uses of formulae that represent an increase in efficiency over parallel contentual methods.

III. A Closer Look at Value

IIIa. Where Does Conductive Use of Formulae Exhibit Its Value?

"Where Concepts Fail ..."

"Where Concepts Fail ..."

"Thus even where concepts fail, there appears a word just in the nick of time."

Goethe, *Faust* I (Mephistopheles)

["Denn eben wo Begriffe fehlen, Da stellt ein Wort zur rechten Zeit sich ein."]

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IIIb. What is Failure?

What is Failure? I

Q: What is it for a concept to fail?

Dominant 19th century answer. . . and, in an important sense, Hilbert's answer too.

A: When there is no item given in intuition from which it may be derived in a respectable ($\approx$ empiricist?) way—that is, by a process of abstraction.

Being derivable in this way keeps a "representation" from being a mere figment of thought.

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What is Failure? II

Where there are not genuine concepts but only signs, there is no genuine reasoning as traditionally conceived.

For genuine reasoning consists in a logical ordering of genuine judgments.

And the logical ordering of judgments is in the first instance a logical ordering of their *contents*, which are genuine propositions.

And, generally speaking, genuine propositions (in this sense) are relations between concepts.

So, where there are not genuine concepts, there are not genuine propositions . . . and where there are not genuine propositions, there are not genuine judgments . . . and where there are not genuine judgments, there is not genuine reasoning.

Where concepts fail, then, there is not genuine reasoning.

This is the significance of failure of concepts.

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IIIc. Is Failure Bad?

Some have thought so . . .

Schopenhauer's View

"...concepts derive their content from the intuitive realm and therefore the entire structure of the world of thought rests upon intuitions. We must therefore be able to go back from every concept, even if indirectly through intermediate concepts, to the intuitions from which it is itself abstracted ... That is to say, we must be able to support it with intuitions which stand to the abstractions in the relation of examples. ...

These intuitions ... afford the real content (*realen Gehalt*) of all our thought, and whenever they are wanting *we have not had concepts but mere words in our heads (blosse Worte im Kopfe)*. In this respect our intellect is like a bank which holds notes (*Zettelbank*), which, if it is to be sound, must have cash in its safe, so as to be able to meet all the notes it has issued, in case of demand; the intuitions are the cash and the concepts the notes."

The World as Will and Representation (1844), vol. 2, ch. 7 (cf. *Sämtliche Werke*, vol 2, 76 for German version)

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Weyl's Statement

In a similar vein, Weyl remarked:

"An existential proposition (*Existentialsatz*)—something like "there is an even number"—is not at all a judgment in the genuine sense of an assertion of a fact. Existential facts are an empty invention (*Erfindung*) of the logicians. "2 is an even number": that is a real (*wirkliches*), a factual (*Sachverhalt*) expression of a given judgment; "there is an even number" is only a judgment-abstract (*Urteilsabstrakt*) obtained from this judgment. Just as I take knowledge (*Erkenntnis*) to be a valuable (*wertvollen*) holding (*Schatz*), so I regard the judgment-abstract as paper (*Papier*), which somehow indicates a holding without saying where it is. Its only value can lie in its ability to get me to search for the holding. The paper is worthless so long as it is not realized (realisiert) by a genuine (wirkliches) judgment such as "2 is an even number" that stands behind it."

"Über die neue Grundlagenkrise der Mathematik (1921), *Mathematische Zeitschrift* 10, 54

IIId. Is Failure Bad?

Others have thought "no" ...

"Where Concepts Fail . . .": Leibniz

Leibniz . . .

". . . I hold [expressions concerning infinites, MD] to be mental fictions (*pro mentis fictionabus*), suited for use in calculations, like the imaginary roots in algebra. And yet I have demonstrated that these expressions (*expressiones*) are a great aid in shortening thought (*ad compendium cogitandi*) and also in discovery (*ad inventionem*), and it is not possible that they should lead us into error (*in errorem ducere non posse*)."

Letter to Fr. Des Bosses, March 17, 1707 (reprinted in Latin original in J. E. Erdmann (ed.), *Gottfried Wilhelm Leibniz: Opera Philosophica* (1840), 436

- Where concepts fail in mathematics, there are non-semantical uses of signs to take their place
- These uses have value as calculating devices—like that which characterizes the use of imaginary roots in algebra
- In particular, they can be shown to increase the efficiency of conclusory reasoning . . . both for previously believed and new conclusions
- And they can do so without sacrificing justificative quality

"Where Concepts Fail ...": Lambert

Lambert ...

"No one has yet formed himself a clear representation of all the members of an infinite series, and no one is going to do so in the future. But we are able to do arithmetic with such series, to give their sum, and so on, by virtue of the laws of symbolic knowledge. We thus extend ourselves far beyond the borders of our actual (*wirklichen*) thinking. The sign $\sqrt{-1}$ represents an unthinkable non-thing. And yet it can be used very well in finding theorems. What are usually regarded as specimens of the pure understanding can be viewed most of the time as specimens of symbolic knowledge."

Letter to Kant, October 13, 1770

- Where concepts fail in mathematics, there are non-semantical uses of signs to take their place
- These are in fact prevalent
- These uses have value as calculating devices—like that which characterizes the use of imaginary roots in algebra
- In particular, they can increase the extent of our knowledge
- ... and also the efficiency with which we attain it?
- ... and they can do so without sacrificing justificative quality?

"Where Concepts Fail . . .": Bernays on Hilbert

"Where concepts fail, a sign is introduced in the nick of time. This is the methodological principle of Hilbert's theory."

Bernays, 'Über Hilbert's Gedanken zur Grundlegung der Arithmetik', *Jahresberichte der deutsche Mathematiker-Verienigung* 30 (1922): 16

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"Where Concepts Fail . . .": Hilbert on Hilbert

". . . just as the infinite, in the sense of the infinitely small and the infinitely large, can . . . be shown to be a mere way of speaking (*eine bloße Redensart*), so we must also recognize the infinite in the sense of the infinite totality (*unendlichen Gesamtheit*) as something merely apparent (*bloß Scheinbares*) when we encounter it as a means of inference (*in den Schlußweisen vorfinden*)." †

"On the Infinite" (1926), 162

†: Here Hilbert has specifically in mind the use of unbounded existential quantification over infinite domains in inferences. Recall Weyl's "Existential facts are an empty invention (*Erfindung*) of the logicians."

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IV. Signs to the Rescue?

Signs to the Rescue?

Can use of signs in conclusory reasoning preserve reasoning as a means of securing genuine contentual belief?

The answer of those supporting conductive use of formulae has been "Yes".

How do they defend this view?

Generally speaking, by some type of metamathematical evaluation.

That is, by showing that whenever conclusory use of signs results in a sentence that has genuine content, this content has the sanction(s) we desire our beliefs to have (e.g. that they're true, justified by certain means, etc.)

This recalls an important distinction Hilbert drew between the failure of concepts in mathematics (not bad), and the failure of concepts in metamathematics (bad).

In metamathematics he accepted the traditional view of failure of concepts.

Signs to the Rescue? Yes, but Intuition is Needed Too . . .

". . . abstract operation with . . . contents (*Inhalten*) has proved . . . inadequate and uncertain. So, as a precondition for the application of logical inferences and for the activation (*Betätigung*) of logical operations, something must already be given in representation (*in der Vorstellung*): certain extra-logical discrete objects, which exist intuitively as immediate experience prior to all thought. . . . I find the objects of the theory of numbers in the signs themselves . . . Here lies the firm philosophical orientation which I require as requisite to the grounding of pure mathematics, and to all scientific thinking, understanding, and communication. 'In the beginning,' we may say 'was the sign'.†"

Hilbert, "Neubegründung der Mathematik" (1922), 162–163 (reprint in *Ges. Abh.* III)

†: "Am Anfang . . . ist das Zeichen." This is a reference to Mephistopheles' "Im Anfang war die Tat" (*Faust*), or "In the beginning was the Act." This suggests that Hilbert may have had Brouwer's counter-veiling "Mathematics is created by a free action independent of experience . . ." ("On the foundations of mathematics" (1907), 97) in mind.

V. Signs to the Rescue

Signs to the Rescue . . .

“. . . it is a received opinion, that language has no other end but the communicating [of] our ideas, and that every significant name stands for an idea. . . . a little attention will discover, that it is not necessary (even in the strictest reasonings) [that] significant names which stand for ideas should, every time they are used, excite in the understanding the ideas they are made to stand for: in reading and discoursing, names being for the most part used as letters are in *algebra*, in which though a particular quantity be marked by each letter, yet to proceed right it is not requisite that in every step each letter suggest to your thoughts, that particular quantity it was appointed to stand for. . . . the communicating of ideas marked by words is not the chief and only end of language, as is commonly supposed.”

Berkeley, *A Treatise concerning The Principles of Human Knowledge* (1710), 37 (emphasis Berkeley’s)

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The Separation of Reasoning from Content I

In Berkeley's view, there is a broad type of reasoning that does not take the form of a sequence of genuine judgments.

Rather, it takes the form of a sequence of actions which, at their end, can be interpreted (and evaluated) in such a way as to yield judgment.

For this to be, Berkeley noted, it is not necessary that language always be used contentually in reasoning.

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The Separation of Reasoning from Content II

"EUPHRANOR. Words, it is agreed, are signs: it may not therefore be amiss to examine the use of other signs, in order to know that of words. Counters, for instance, at a card-table are used, not for their own sake, but only as signs substituted for money, as words are for ideas.

. . . [I]s it necessary every time these counters are used throughout the progress of a game, to frame an idea of the distinct sum or value that each represents?

ALCIPHRON. By no means: it is sufficient the players at first agree on their respective values, and at last substitute those values in their stead.

EUPHRANOR. And in casting up a sum, where the figures stand for pounds, shillings, and pence, do you think it necessary, throughout the whole progress of the operation, in each step to form ideas of pounds, shillings, and pence?

ALCIPHRON. I do not; it will suffice if in the conclusion those figures direct our actions with respect to things."

Alciphron, Dialogue VI

Something very much like this view of language seems to have been behind Hilbert's view of the conductive use of formulae.

The Value in Avoiding Contentual Reasoning

It may be, then, that there are legitimate non-contentual forms of reasoning.

Is there any particular value in it?

This is the question we have been pursuing today, and it seems the answer is "yes".

Basically, the answer is "conservation of resources."

Roughly, we stand to get more out of our limited investigative resources if we take advantage of the right opportunities for conductive use of formulae.

Let's close by looking at a couple of statements of this basic idea.

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The Value of Non-Contentual Reasoning: Hilbert

"To make it a universal requirement that each individual formula be interpretable by itself is by no means reasonable; on the contrary, a theory by its very nature is such that we do not need to fall back upon intuition or meaning in the midst of some argument. What the physicist demands precisely of a theory is that particular propositions be derived from laws of nature or hypotheses solely by inferences, hence on the basis of a pure formula game, without extraneous considerations being adduced. Only certain combinations and consequences of the physical laws can be checked by experiment—just as in my proof theory only the real propositions are directly capable of verification."

Hilbert, "Grundlagen der Mathematik" (1928), 15

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The Need for Non-Contentual Reasoning: Mach

"We must admit, therefore, that there is no result of science which in point of principle could not have been arrived at wholly without methods. But, as a matter of fact, within the short span of a human life and with man's limited powers of memory, any stock of knowledge worthy of the name is unattainable except by the greatest mental economy. Science itself, therefore, may be regarded as a minimization problem, consisting of the completest possible presentment of facts with the least possible expenditure of thought."

Mach, *The Science of Mechanics*, 585

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The Need for Non-Contentual Reasoning: Thomas Hill

"The most striking character of the written language of algebra and of the higher forms of the calculus is the sharpness of definition, by which we are enabled to reason upon the symbols by the mere laws of verbal logic, discharging our minds entirely of the meanings of the symbols, until we have reached a stage of the process where we desire to interpret our results. The ability to attend to the symbols, and to perform the verbal, visible changes in the position of them permitted by the logical rules of the science, without allowing the mind to be perplexed by the meaning of the symbols until the result is reached which you wish to interpret, is a fundamental part of what is called analytical power. Many students find themselves perplexed by a perpetual attempt to interpret not only the result, but each step of the process. They thus lose much of the benefit of the labor-saving machinery of the calculus and are, indeed, frequently incapacitated for using it."

Thomas Hill (1818–1891), "The Uses of Mathesis", *Bibliotheca Sacra* 32 (1875), 505

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Conclusions

The value of non-contentual reasoning was seen as residing in its efficiency.

The basic idea is that contentual thinking makes a heavier call on scarce resources than does conductive use of formulae.

The upshot is that if we are to attain desirable efficiency in our reasoning, we must minimize its contentuality.

In other words, to attain desirable efficiency in our reasoning, we must utilize our capacity for conductive use of formulae.

This raises various questions of course. For example:

- Does the efficiency of conductive uses of formulae lie in one thing, or in different things?
- Is there a tolerably precise description of any of these?
- Is there an even approximate way to measure these efficiencies?

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