In Search of Formalism

Lecture IV: Conductive Use of Formulae: Efficiency, Rigor & Reliability

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I. Review/Preview

Review/Preview I
Previous Statement of Formalism

We arrived at the following (still provisional) formulation of formalism:

There is extensive conductive use of formulae in mathematical reasoning that is both (i) valuable as a means of improving epistemic efficiency, and (ii) reliable as a means of determining what to believe.

We noted last time that this formulation of Formalism "fails" to capture an important element of the traditional view of the value of formalization—namely, that of achieving rigor.

This omission was not accidental. Nor do I view it as a failure.

The reason is that there is nothing distinctively Formalist in the idea that formalization is a means of achieving rigor.

Formalists and non-formalists alike can and commonly have maintained such a view.

I would also note that to use formalization as a means towards rigor is more a representative than a conductive use of formulae, though this is complicated.

1 $\longrightarrow$

Review/Preview II
Efficiency: Examples & Descriptions

We considered several proffered examples of efficiency-producing devices, and general descriptions of the nature of the efficiency produced.

- Introduction of complex magnitudes in algebra

  "a great aid in shortening thought, and also in discovery" (Leibniz)

  "simplify the theorems on the existence and number of the roots of an equation" (Hilbert)

- Introduction of elements "at infinity" in projective geometry

  "mak[es] the system of the laws of connection as simple and perspicuous as possible" (Hilbert)

  "by the principle of duality, fifty detailed proofs may suffice to establish as many as a hundred theorems" (Coxeter)

- Adjunction of the so-called ideal propositions to the finitary propositions in logic

  "maintain[s] the formally simple rules of ordinary Aristotelian logic" (Hilbert)

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Review/Preview III
The Problem of Value

Phrases such as "shorten thought", "simplify theorems", "simplify system of laws", "make laws perspicuous" and "formally simplify rules" may be suggestive, but they are not very clear.

Coxeter's "fifty detailed proofs may suffice to establish as many as a hundred theorems" is more exact, but there are many complications both of detail and of principle.

The unclarity of so many of the characterizations of the efficiency produced by conductive use of formulae is problematic, and for two reasons.

The first is that the claims of efficiency are widespread and persistent, among formalists and non-formalists alike.

They therefore seem to reflect the considered experience of practicing mathematicians.

And while this is hardly proof of their correctness, it is a reason to take them seriously enough to try to clarify and evaluate.

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Review/Preview IV
The Problem of Value

The second reason is that . . .

- conductive use of formulae is the core element of formalism

  "The essence of formalism is that we *use* symbols according to definitely prescribed rules, not that we talk about symbols."

  H. B. Curry, "Mathematics, Syntactics and Logic", *Mind* 62 (1953), 175 (emphasis Curry's)

  $\therefore$

- convincing evaluation of formalism, positive or negative, depends on convincing treatment of the value of conductive use

  $\therefore$

- convincing evaluation of formalism depends on convincing treatment of efficiency

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Review/Preview V
Is There a Convincing Treatment of Efficiency?

Overall, the answer to the question above seems to be “Not at present”.

Part of the difficulty is that efficiency seems to take different forms.

simplifying effects of duality in projective geometry $\not\approx$
simplifying effects of complex magnitudes in algebra $\not\approx$
simplifying effects of ideal sentences in logic $\not\approx$
simplifying effects of number ideals in number theory $\not\approx$ ...

So, there is little reason to think that there is a single type of efficiency that constitutes the value of all the different types of conductive uses of formulae there are in mathematics.

This notwithstanding, various thinkers in the late 19th and early 20th centuries offered a proposal for a unified treatment of efficiency.

We briefly considered this proposal in the third lecture, and we'll return to it now for a closer look.

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II. A Unified Account of Efficiency: Efficiency Through Abstraction

Efficiency Through Abstraction

Let's call this unified conception of efficiency the *Efficiency Through Abstraction* (*ETA*) conception.

Following suggestions of its advocates, we can state it as follows:

ETA: A prime means of attaining efficiency in any type of mathematical reasoning is to

i. abstract away from the meanings or contents of the terms appearing therein,

   and to

ii. replace judgment and inference based on the meanings of those terms with judgments and inferences concerning syntactical properties of the expressions themselves and the application of a system of syntactically expressed (or expressible) rules of procedure to them.

This was a popular idea in the 18th, 19th and 20th centuries.

An interesting and unusually detailed statement of it was given by the American mathematician, and one time president of Harvard University, Thomas Hill, in the late 19th century.

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Hill on Abstractive Use of Language in Mathematics I

"... for the highest mathematical success there is a ... power requisite, and that is the power of using the language appropriate to ... mathematics. ... The syntax of this language is energetic and forcible,—the symbols by which it is written are marvels of condensation. The most striking character of the written language of algebra and of the higher forms of the calculus is *the sharpness of definition*, *by which we are enabled to reason upon the symbols by the mere laws of verbal logic*, *discharging our minds entirely of the meanings of the symbols*, until we have reached a stage of the process where we desire to interpret our results."

"The Uses of 'Mathesis' ", *Bibliotecha Sacra & Theological Eclectic* 32 (1875), 504–505

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Hill on the Advantages of Abstraction I

The key ideas are

(a) that it is possible to make declarations of the features of concepts that figure in reasoning concerning them so explicit and complete that these declarations come to *syntactically constitute*† that which can be non-inferentially asserted of the concepts

(b) that use of such declarations in reasoning then become essentially conductive uses of formulae

and

(c) that such use of formulae "discharges our minds entirely of the meanings of the symbols, until . . . we desire to interpret our results."

†: In saying that a declaration d *syntactically constitutes* all that can be non-inferentially asserted of a concept $\mathfrak{c}$, I mean that any assertion whose syntax is not that of d is not admissible as a non-inferred assertion about $\mathfrak{c}$.

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Hill on the Advantages of Abstraction II

Hill believed this "discharging" to be advantageous because in the *process* of reasoning

• it decreases perplexity (because we're generally less likely to be perplexed by the syntactical features of symbols than by their meanings)

• it decreases labor (by decreasing the perplexity of the judgments composing it)

• it helps avoid incapacitation as a reasoner (owing to sometimes unmanagable perplexity at the level of meaning)

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Hill on the Advantages of Abstraction III

In Hill's own words . . .

"The ability to attend to the symbols, and to perform the verbal, visible changes in the position of them permitted by the logical rules of the science, without allowing the mind to be perplexed by the meaning of the symbols until the result is reached which you wish to interpret, is a fundamental part of what is called *analytical power*. Many students find themselves perplexed by a perpetual attempt to interpret not only the result, but each step of the process. They thus lose much of the benefit of the labor-saving machinery of [a] calculus and are, indeed, frequently incapacitated for using it."

"The Uses of 'Mathesis"', *Bibliotecha Sacra & Theological Eclectic* 32 (1875), 505

• syntactical use of symbols during the process of reasoning, is a fundamental reason why analytic method is so powerful

• • it decreases perplexity, the threat of which is serious

• it decreases the labor of reasoning (by decreasing the extent to which it has to cope with perplexity)

• it helps avoid incapacitation due to unmanageable perplexity at the level of meaning, also a serious problem

Hill on the Need for Abstraction I

Hill went on to describe "a curious example of this incapacity, shown, about thirty years ago, by successive classes of sophomores in Harvard College."

The "incapacity" concerned inability to master the proof of Arbogast's Polynomial Theorem.

This is a theorem that gives a method of finding, for a function ϕ that can be developed in a power series, the coefficient of the next x^n in the development of

$$\phi(a + bx + cx^2 + dx^3 + \ldots)$$

Hill and fellow instructors noticed a pronounced inclination among sophomores at Harvard to want to keep the developing *quantities "in mind"* while attempting to prove the theorem, and resisted proceeding syntactically and interpreting only at the end.

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Hill on the Need for Abstraction II

According to Hill, these students had great difficulty arriving at anything they recognized as a proof, and they had little if any retention.

He saw this as concrete illustration of the importance of conductive use of formulae.

But Hill's basic point seems right—those who conductively used formulae made some type of gain that those "incapacitated" by their contentual approach did not.

Moreover, this gain seems to have been an epistemic gain of some sort.

There are, of course, serious questions concerning what it was that this alleged gain consisted in. But whatever it was seems better than epistemic incapacitation or paralysis.

Hill on the Conductive Use of Symbols

Hill's views on the value of conductive use of formulae seem to have been rooted in a Berkeleyan conception of language.

Like Berkeley he emphasized the use of language as a logistic instrument.

"Language is not only a means of recording the results of our thinking; it is an instrument of thought, and that of the highest value. I . . . here refer . . . to the fact that . . . when we have definitely fixed the meaning of a word or symbol, so that it shall include neither more nor less than we intend, we can discharge our minds of that meaning, and operate upon the symbol as a symbol, by the mere rules of the syntax of the language and the laws of logic, and be confident that every result thus attained upon the symbols as such, will, when properly interpreted, give a truth concerning the things originally symbolized."

Hill, "The Uses of 'Mathesis"' (1875), 506

Similarities with Leibniz, Lambert and Hilbert hardly need pointing out.

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Hill's Views: Summary I

Hill offered not inconsiderable views of

(i) the conditions under which conductive use of formulae is legitimate or justifiable,

(ii) the nature of the efficiency represented by conductive use of formulae, and

(iii) its overall value

In connection with (i), his view was that . . .

Conductive use of formulae in reasoning is justified when the elements of meaning that would be appealed were a given body of reasoning to be contentually conducted have been so explicitly and fully declared that no matter how one might interpret the syntax of these declarations, conclusions drawn from them would always be true provided they were.

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Hill's Views: Summary II

(ii) the nature of the efficiency represented by conductive use of formulae, and

(iii) its overall value

Concerning (ii), his thinking seems to have been along the following lines.

Judgment concerning the contents of mathematical statements and inference that proceeds in terms of it is generally less easy to arrive at (i.e. it is more halting) and less certain as to truth/validity than judgment/inference concerning their syntactical properties and whether given rules apply to them.

He seems to have seen the overall value of conductive use as following from this—that value being that fewer investigative resources are generally required for conductive use of formulae than are required for the parallel conductive use of contents.

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Efficiency Through Abstraction: Conclusions

Neither Hill nor anyone else (e.g. Leibniz, Berkeley, Lambert or Hilbert) has proven that conductive use of formulae generally improves efficiency.

This notwithstanding, there is evidence for this view. This includes:

- that syntactical manipulation or something very much like it plays a significant role not only in arithmetical computation but in mathematical reasoning more generally

  and

- that there appears to be a significant notion(s) of "difficulty" with respect to which the following are generally true:
  - past a certain basic point, the difficulty of grasping the syntactical form of a mathematical statement is not as great as that of grasping its meaning
  - past a certain basic point, the difficulty of determining of a given sentence that its syntactic form matches (or does not match) that of another, or that it satisfies a given syntactical schema is not as great as that of determining that that sentence is true under an assumed or otherwise given interpretation

Efficiency Through Abstraction: Conclusions

Claim I: This evidence is strong enough to warrant the further investigation of the possible efficiencies represented by conductive uses of formulae.

Claim II: It warrants consideration of how such efficiency phenomena might stand to affect the plausibility and viability of formalism as a philosophy of mathematics.

Formalism

There is extensive conductive use of formulae in mathematical reasoning that is:

(i) reliable as a means of determining what to believe,

and

(ii) valuable as a means of making mathematical thinking more efficient.

We'll now turn to a closer look at Claim II.

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III. Efficiency & Reliability

Efficiency Through Abstraction: Conclusions

It may have struck you as strange or even misguided to have given so much attention to the question of efficiency when, as everyone knows, it is the reliability component of Formalism that is most problematic.

Indeed, you might think, it's so problematic that the other components really don't matter.

I'll now briefly argue that the two issues are linked in a critical way, a way that suggests that Gödel's Second Incompleteness Theorem (G2)—commonly seen as the main obstacle to satisfactory treatment of reliability—may not have the effect it has commonly been taken to have.

Formalism

There is extensive conductive use of formulae in mathematical reasoning that is:

(i) reliable as a means of determining what to believe,

and

(ii) valuable as a means of making mathematical thinking more efficient.

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The Dilution Problem

Formalism seeks to preserve the efficiency it sees in the conductive use of formulae.

It sees this efficiency as *epistemic* efficiency.

Specifically, it sees it as a means of increasing the extent of belief per unit investigative resource consumed without sacrificing its quality.

To accomplish this, it must

1. provide a means of converting conductive uses of formulae into justifications,

   and

2. it must do this in such a way that the quality of the resulting justification is generally as high as that of the contentual reasoning it's intended to replace.

Conductive use of formulae will not be overall epistemically valuable if the gains it affords in increased extent of belief are negated by a corresponding drop in the general quality of their justification.

This is what I call the Dilution Problem.

The Dilution Problem

How is the Dilution Problem to be managed?

The only systematic strategy seems to be to provide a justification of the general reliability of conductive use of formulae that is of the same basic quality as that of the contentual reasoning it is intended to replace.

This then brings us to a larger and more difficult problem facing the Formalist of which the Dilution Problem is but one aspect.

This is what I call the *Division Problem*.

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The Division Problem

This is the problem of dividing the reasoning in mathematics that constitutes conductive use of formulae (call it $\mathfrak{I}$) from contentual mathematical reasoning in such a way that:

I. Efficiency: There are sufficient means in $\mathfrak{I}$ to guarantee a significant gain in efficiency over what would be the case were the results established by it to be established by contentual reasoning (call this latter body of contentual reasoning $\mathfrak{R}$),

and

II. Non-Dilutive Reliability: The justificative quality of our evidence for the reliability of $\mathfrak{I}$ is on a par with that which is characteristic of $\mathfrak{R}$.

As indicated earlier, the common worry has been that G2 bars satisfaction of Non-Dilutive Reliability.

Let's now consider the evidence for this common view.

The Formalist's Task Concerning Reliability

Formalist's Task Concerning Reliability:

Prove the reliability of $\mathfrak{I}$ by means having the same justificative quality as is characteristic of proofs in $\mathfrak{R}$.

Since $\mathfrak{I}$ can't be reliable without being consistent the Formalist cannot carry out this task unless she can carry out the following:

Consistency Task: *Prove the consistency of $\mathfrak{I}$ by means having the same justificative quality as those in $\mathfrak{R}$.*

It is here that the Formalist meets G2.

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The Consistency Task I

To save on technical details, I'll not give an exact statement of G2, but move directly to the type of "interpretation" of it that is commonly believed to thwart the Formalist.

I'll call this statement "G2+" because, in addition to a version of G2, it requires a number of sometimes problematic interpretive assumptions.†

G2+: If (i) $\mathfrak{I}$ is a recursively axiomatizable§ consistent extension of $\mathfrak{R}$, (ii) $\mathfrak{R}$ weakly represents all recursively enumerable (r.e.) sets,‡ and (iii) $Con_{\mathfrak{I}}$ is a formula that 'expresses' the metamathematical proposition that $\mathfrak{I}$ is consistent, then $\nvdash_{\mathfrak{I}} Con_{\mathfrak{I}}$ (hence $\nvdash_{\mathfrak{R}} Con_{\mathfrak{I}}$).

†: For example, that the "provability" formula $Pr_{\mathfrak{I}}(x)$ in terms of which $Con_{\mathfrak{I}}$ is formulated expresses the notion of provability-in-$\mathfrak{I}$ only if it satisfies that set of conditions generally known as the *Derivability Conditions*.

§: A theory T is recursively axiomatizable iff there is a recursive set α of theorems of T such that T it is the deductive closure of α. The set of theorems of a recursively axiomatizable theory is recursively enumerable.

‡: A set of numbers θ is weakly representable in a theory T just in case there is a formula $\mathcal{F}(x)$ of T such that for every number n, $n \in \theta$ iff $\vdash_T \mathcal{F}(\mathbf{n})$, where $\mathbf{n}$ is the numeral (or other canonical term) in T for n.

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The Consistency Task II

Does G2+ imply that the consistency task for $\mathfrak{I}$ cannot be carried out?

The answer, not surprisingly, is "It depends ..."

It depends on a number of things, one of which will be my focus today.

This is the relationship between $\mathfrak{I}$ and those derivations in $\mathfrak{I}$ that give it whatever improvement in efficiency over $\mathfrak{R}$ it enjoys.

For convenience, I'll let $\mathfrak{I}_{\Uparrow}$ stand for the set of derivations in $\mathfrak{I}$ that constitute improvements in efficiency over $\mathfrak{R}$. I'll call $\mathfrak{I}_{\Uparrow}$ the *efficiency reduct* of $\mathfrak{I}$.

[To say of a derivation D in $\mathfrak{I}$ of a sentence S that it constitutes an improvement in efficiency over $\mathfrak{R}$ is to say that it is more efficient than *any* proof in $\mathfrak{R}$ of S ... however efficiency is conceived and measured.]

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The Consistency Task III

So, does G2+ imply that the Formalist's consistency task cannot be carried out for $\mathfrak{I}$ if $\mathfrak{I}$ is a theory for which G2+ holds?

The answer this time is "No, at least not generally."

It depends, in particular, on the relationship between $\mathfrak{I}$ and $\mathfrak{I}_{\Uparrow}$.

If $\mathfrak{I}_{\Uparrow}$ is a strict subtheory of $\mathfrak{I}$, the fact that G2+ holds of $\mathfrak{I}$ does not imply that it also holds of $\mathfrak{I}_{\Uparrow}$.

Hence, in such cases, the fact that the consistency task for $\mathfrak{I}$ might not be executable does not imply that the consistency task for $\mathfrak{I}_{\Uparrow}$ is not.

It will depend . . .

The Consistency Task: Conclusion

Is the distinction between a theory and its efficiency reduct significant?

Certain facts suggest that it may be. These include the facts that:

A. For the average familiar formal system (e.g. *PA*, PA^2, *ZF*, etc.) we are far from knowing what any of its efficiency reducts would look like. In particular, we don't know whether or not it is a strict subtheory of the system in question.

B. Some of these theories have significant capacities to prove the consistency of important families of their subsystems. *PA* and *ZF*, for example, are *reflexive*—that is, they prove the consistency of each of their finitely axiomatizable subtheories.

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In Search of Formalism: Conclusion

I hope that what I've said in these lectures has shed at least a little light on the question of what Formalism is.

I hope also to have given a more exact sense of the challenges that confront anyone who wants to know whether Formalism is a viable philosophy of mathematics.

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