Ideals of Proof

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Type I Ideals: Local

The term 'ideals' in the title has two broadly different senses.

I. Virtues, requisites or perfections of individual proofs, of beliefs based on them or of whole systems of proofs and beliefs.

On the local or individual side, these have traditionally included such properties as:

- rigor*
- certainty (of justifications provided by proofs and of beliefs based on them)
- explanatory capacity (of proofs and the justifications provided by them)
- efficiency (of the processes of finding and/or verifying proofs)
- conceptual purity*
- simplicity

List not exhaustive, items not independent.

Type I Ideals: Global

On the global or systemic side, traditional examples include:

- consistency
- irredundancy, independence
- completeness (of various types)
- efficiency (of various types)

List not exhaustive, items not independent.

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Type II Ideals

Type I ideals are thus ideals in the sense that they're ends that ought, in one sense or another, to be pursued by those engaged in the practice(s) of giving proofs.

Type II ideals are ideals in a different sense—a sense in which the ideal in one way or another, and to one extent or another contrasts with the real. Common examples include ...

- Imaginary number $i = \sqrt{-1}$, and its complex cohort $a + bi$ (a, b real numbers)

- Points, lines and planes 'at infinity' in projective geometry

- Kummer's number ideals (i.e. numbers of the form $a_0 + a_1 z + a_2 z^2 + \ldots + a_{n-1} z^{n-1}$ where z is a primitive nth root of unity and a_i an integer)

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Real/Ideal Contrast

Representative statements of this contrast and/or the terms belonging to it include . . .

"Neither the true [positive, MD] nor the false [negative,MD] roots are always real; sometimes they are only imaginary (*seulement imaginaires*); that is, while we can always conceive of as many roots for each equation as I have already assigned†, yet there is not always a definite quantity corresponding to each root so conceived."

Descartes, *La Geometrie* (1637), Bk. III, 175

†: Assigned, that is, according to his statement of the Fundamental Theorem of Algebra.

Real/Ideal Contrast

"Both true and false roots are not always real but on occasion merely imaginary, that is, there may in any equation be conceived to be in imagination as many roots as we mentioned just now, but it may happen on occasion that there is no quantity which accords with those concepts."

Newton, *Observations on Kinckhuysen's Algebra* (ca. 1670): 412–13

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Real/Ideal Contrast

According to Descartes, Newton and others, ideal elements have two characteristic properties:

- they are conceivable

  but

- no quantities correspond to their concepts.

What does “correspond to” mean?

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Real/Ideal Contrast

Traditional answer:

> The concept provides a means of constructing or finding an object that can be apprehended in intuition to satisfy it.

For the case of ideal elements, then, the concept does *not* provide such means.

Traditionally, this was the signal difference between real and ideal elements.

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Real/Ideal Contrast

Representative statements:

"real definition makes known *a priori* the possibility or the genesis of the object defined."

Leibniz, *New Essays ...* (1704), Bk. II, ch. III, §18

"*Real* DEFINITION ... explain[s] the genesis of a thing; that is, how the thing is made or done: as is this definition of a circle, viz., that it is a figure described by the motion of a right line about a fixed point."

Charles Hutton, *A Mathematical and Philosophical Dictionary*(1796)

Real/Ideal Contrast

"*Quantities* are said to be *given* [to exist, MD], which are either exhibited, or may be found."

John Leslie, *Geometrical Analysis ...* (1821), 4

And finally this remark directed explicitly at Descartes' statement ...

"Cartesius called the positive and negative roots of an equation "real", the impossible "imaginary", since the former can be constructed as straight lines, while the latter admit of nothing of the sort."

Otto Stolz, *Grössen und Zahlen* (1891), 17

Real/Ideal Contrast

Core of the Traditional Real/Ideal Distinction

Real Element

(i) Conceivable

(ii) Concept (definition) provides a means of exhibiting an object that can be determined by intuition to satisfy it

Ideal Element

(i) Conceivable

($\sim$ ii) Concept (definition) does NOT provide a means of exhibiting an object that can be determined by intuition to satisfy it

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Ideal Elements: A Tension?

Some have thought there is tension between the two clauses of the traditional conception of an ideal element—that is between

(i) Conceivable

and

($\sim$ ii) Concept (definition) does NOT provide a means of exhibiting an object that can be determined by intuition to satisfy it

Tension: If ($\sim$ ii) holds, (i) can't hold.

Remark: This would imply that (ii) (i.e. the second clause of the traditional characterization of real elements) is implied by (i) and, so, strictly speaking, redundant.

Ideal Elements Are Not Conceivable

Schopenhauer was among these, and he argued as follows:

"... concepts derive their content from the intuitive realm and therefore the entire structure of the world of thought rests upon intuitions. We must therefore be able to go back from every concept, even if indirectly through intermediate concepts, to the intuitions from which it is itself abstracted ... That is to say, we must be able to support it with intuitions which stand to the abstractions in the relation of examples. ...

These intuitions ... afford the real content of all our thought, and whenever they are wanting *we have not had concepts but mere words in our heads*. In this respect our intellect is like a bank, which, if it is to be sound, must have cash in its safe, so as to be able to meet all the notes it has issued, in case of demand; the intuitions are the cash and the concepts the notes."

The World as Will and Representation (1844), vol. 2, ch. 7

Ideal Elements Are Not Conceivable

Distilling a little, we get . . .

Schopenhauer's Syllogism

1. Genuine concepts derive from intuitions which exemplify them.
2. The definitions of ideal elements do not provide intuitions that exemplify them.

$\therefore$

The definitions of ideal elements do not express genuine concepts. They're just words.

Euler's Challenge

Euler presented a somewhat different challenge.

Like Schopenhauer, he, in effect, argued against the conceivability of imaginary numbers.

Unlike Schopenhauer, he didn't ascribe this inconceivability to the failure of definitions of ideal elements to provide exemplifying intuitions for them.

Nor did he claim that ideal elements were inconceivable without qualification.

He only challenged their conceivability *as numbers* or *quantities*.

This seems important because it suggests an alternative to the traditional characterization of the real/ideal distinction. But more on this later ...

Euler's Challenge

"Now as all the numbers which it is possible to conceive, are either greater or less than 0, or are 0 itself, it is evident that we cannot rank the square root of a negative number among the possible numbers, and we must therefore say that it is an impossible quantity. In this manner we are led to the idea of numbers which *from their very nature* are impossible. Those numbers are usually called imaginary quantities, because they exist merely in the imagination."

Euler, *Vollständige Anleitung zur niedern u. höhern Algebra* (1770), §143

Euler's Syllogism

Euler's reasoning—his "syllogism", if you will—was essentially this:

1. Something is a possible number only if we can conceive of it as a number.
2. We can conceive of something as a number only if we can conceive of it as either less than, equal to or greater than 0.
3. We cannot conceive of square roots of negative numbers in this way.

$\therefore$

Square roots of negative numbers are not possible, but "merely imaginary."

Footnote to Euler

Strictly speaking, Euler was wrong to suggest (if he did suggest) that imaginary and complex numbers could not satisfy the following trichotomy principle.

For every x, $x < 0$ or $x = 0$ or $x > 0$.

What can't happen is that the complex numbers satisfy both this and the following principle concerning the products of positive quantities.

For every x, y, if $x > 0$ and $y > 0$, then $x \times y > 0$.

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Central Laws of General Arithmetic not Satisfied by the Complex Numbers

In the complex field, not both of the following laws are satisfiable.

1. Trichotomy: $\forall x(x < 0 \vee x = 0 \vee x > 0)$
2. Products: $\forall x \forall y((x > 0 \wedge y > 0) \rightarrow x \times y > 0)$

Proof:

If (1) were to hold for the complex field, there would be three possibilities for i: (i) $i < 0$, (ii) $i = 0$ or (iii) $i > 0$.

Under alternative (i) $-i > 0$. Hence, by (2), $i^2 > 0$. By the fact that $i^2 = -1$, this implies the absurdity that $-1 > 0$.

Under alternative (ii), $i^2 = 0 \times 0 = 0 = -1$, which is again absurd.

Under (iii), (2) implies that $i \times i > 0$. But since $i \times i = -1$, this implies that $-1 > 0$ ... absurd.

(1) and (2) cannot therefore both hold.

QED

Gauss to the Defense

Euler thus questioned the conceivability of imaginary numbers *as numbers* or *quantities*.

Others, including some with views like Schopenhauer's, questioned their conceivability period more radically.

Gauss famously disagreed with all the above and put forth what he sometimes claimed was a fully adequate defense of imaginary numbers.

The root of his defense was his celebrated "geometrical" interpretation of the imaginaries.

Gauss' Syllogism

Gauss' argument was essentially this:

Gauss' Syllogism

1. The definitions of imaginary numbers do support the construction of intuitively exemplifying objects.
   - [Specifically, they support the construction of "lines" in Gauss' complex plane.]

$\therefore$

2. They satisfy the traditional criterion for 'reality'; specifically, their definitions provide means of exhibiting objects that can be determined by intuition to satisfy them

$\therefore$

It's wrong to call them "imaginary", "impossible" or "ideal." They're as real as anything in mathematics.

Gauss' Recommendation

In Gauss' view, then, the traditional real/ideal distinction ought to be abandoned, at least with respect to the quantities dealt with in analysis.

More precisely, (i) *imaginary* or ideal elements ought to be retained (because of their usefulness) while (ii) the real/ideal distinction, and the accompanying terminology, ought to be abandoned.

Finally, a new terminology of "possible quantities" ought to be adopted and applied to both traditionally real quantities and traditionally imaginary quantities.

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Gauss' Larger Viewpoint

Interestingly, the rudiments of this recommendation were presented in Gauss' dissertation.

"If imaginary quantities are to be retained in analysis (which for many reasons seems better than to abolish them, provided they are established on a sufficiently solid foundation) it is necessary that they be considered as equally possible with real quantities, on which account I should prefer to include both real and imaginary quantities under the common designation *possible quantities*."

Demonstratio nova theorematis . . . (1799), 6 (*Werke*, vol. 3)

In 1831 he provided what he believed was a "sufficiently solid foundation"—namely, his 'geometrical' interpretation.

In the same footnote, he declared, without explanation, that it would be "laughable" to try to establish the reality of the concept "rectilinear right equilateral triangle." Hmmm?

The Geometrical Interpretation

“. . . the arithmetic of the complex numbers is capable of the most intuitive sensible representation (der anschaulichsten Versinnlichung) . . . By this representation (Darstellung) the effects of the arithmetical operations on the complex quantities becomes capable of sensible representation that leaves nothing to be desired.

. . . in this way the true metaphysics of the imaginary quantities is placed in a bright new light.”

Gauss, *Theoria residuorum biquadraticorum, commentatio secunda* (Anzeige) (1831), 174–75 (*Werke* 2)

Gauss' Impurity

Not everyone agreed that Gauss' interpretation provided a "sufficiently solid foundation" for imaginary numbers.

A common reaction was that Gauss' interpretation was insufficient in this respect: it didn't found the imaginary numbers *as numbers*.

Rather, it presented them as geometrical or quasi-geometrical items , and that was not enough to secure a rightful place for them in arithmetic.

Frege on Gauss' Impurity

These points were made by a number of mathematicians in the 19th and 20th centuries.

Frege, for example, had this to say about it:

"The overcoming of [the] reluctance [to accept imaginary numbers] was facilitated by geometrical interpretations; but with these, something foreign was introduced into arithmetic. Inevitably there arose the desire to . . . extrude these geometrical aspects. It appeared contrary to all reason that purely arithmetical theorems should rest on geometrical axioms; and it was inevitable that proofs which apparently established such a dependence should seem to obscure the true state of affairs. The task of deriving what was arithmetical by purely arithmetical means, i.e. purely logically, could not be put off.

Theories of Formal Arithmetic (1886), 116–17 (*Collected Papers*)

Gauss' Impurity as a Motive for Logicism

Frege for one, then, didn't accept Gauss' foundation for the complex numbers. Specifically, he rejected it because of its appeal to 'foreign'—that is, geometrical—elements.

And the inadequacy of Gauss' interpretation was a prime motive for his logicism—a program to provide a purely arithmetic foundation for arithmetic (including complex arithmetic).

Gauss' Impurity

The American mathematician H. B. Fine similarly remarked on the 'impurity' of Gauss' attempted foundation . . . though he did not agree with Frege's proposed logicist correction, and recommended a formalist correction instead.

"A reality [Gauss' geometrical interpretation] has thus been found to correspond to the hitherto uninterpreted symbol $a + ib$. But this reality has no connection with the reality which gave rise to arithmetic . . . and does not at all lessen the purely symbolic character of $a + ib$ when regarded from the standpoint of that reality, the standpoint which must be taken in a purely arithmetical study of the origin and nature of the number concept.

The connection between the numbers $a + ib$ and the points of a plane is purely artificial."

The Number-System of Algebra (1902), 44–45

Gauss' Impurity

Finally, Gauss himself, within a few years, seemed to lose some of the early enthusiasm he'd shown for his interpretation.

"The representation (Darstellung) of the imaginary quantities as relations of points in the plane is not so much their essence (Wesen) itself, which must be grasped in a higher and more general way, as it is for us humans the purest or perhaps a uniquely and completely pure example of their application."

Letter to Max Drobisch, August 14, 1834 (*Werke* 10, 106)

Here Gauss seems to concede that his geometrical interpretations *can't* be derived from the imaginary arithmetical concepts themselves, but are only "examples" of their application—albeit "uniquely and completely pure" examples.

Upshot

Gauss' interpretation did not do what it purported to do—namely, show that imaginary *numbers* are real, according to the standard of reality set out in the traditional real/ideal distinction.

It failed because it didn't show how to exhibit the concept or definition of an imaginary number *as a number*, but only as a quasi-geometrical entity of some sort.

In the argument for this failure, we see one example of an interaction between ideals in the two senses distinguished at the outset—namely, the Type I ideal of purity, and the Type II ideal of imaginary numbers.

Upshot

We see another point of connection if we consider more carefully the reasons for wanting real rather than ideal concepts or definitions.

Traditionally, there have been two main reasons:

- To secure rigor of proofs (i.e. reliability of theorems).
- To meet broad skeptical challenges aimed at undermining the very undertaking of proof.

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Exhibition for the sake of Rigor and Reliability

"... when the property brings us to see the possiblity of a thing it makes the definition real, and as long as one has only a nominal definition he cannot be sure of the consequences he draws, because if it conceals a contradiction or an impossibility he would be able to draw the opposite conclusions."

Leibniz, *Discourse on Metaphysics*, §XXIV

"If ... demonstration is to be rigorous, the possibility must be demonstrated *a priori*. Clearly, we cannot safely devise demonstrations about any concept, unless we know that it is possible; for of what is impossible, i.e. involves a contradiction, contradictories can also be demonstrated. This is the *a priori* reason why possibility is required for real definition."

Leibniz, *Of Universal Synthesis & Analysis*, 13

Exhibition to Justify the Undertaking of Proof

And, speaking of the ordering of the first four propositions of Book I of the *Elements*, Proclus writes:

"... our geometer [Euclid, Bk. 1] follows up these problems [Props. I–III] with his first theorem [Prop. IV] ... For unless he had previously shown the existence of triangles and their mode of construction, how could he discourse about their essential properties? Suppose someone ... should say: "If two triangles have this attribute, they will necessarily also have that." Would it not be easy ... to meet this assertion with "Do we know whether a triangle can be constructed at all?" ... It is to forestall such objections that the author of the *Elements* has given us the construction of triangles ... *These propositions are rightly preliminary to the theorem ...*"

Proclus (ca. 410–485), *Commentary*, 182–183

A Parallel with the Law

Legal proceedings are also concerned with the reliability or verdicts and also with the justification of undertaking proof (i.e. of going to trial).

Some light may be shed by considering a possible parallel between exhibition in mathematics and the principle of *corpus delicti* from western jurisprudence.

Corpus Delicti

- Literally, "the body of crime."
- Example: In murder, *corpus delicti* has two components:

  i. death as a result of an act (and nothing says death quite like a *corpse* !)

  and

  ii. the criminal agency of another as its means

- *Corpus delicti* is thus concerned primarily with proper standards for proving the "existence" of a crime.
- Establishing the "existence" of a crime is a precondition not only of legitimate *conviction* but of legitimate *trial* as well.

Corpus delicti is a principle concerning the legitimate *undertaking* of proof.

Construction & Corpus Delicti: A Parallel?

The thinking behind *corpus delicti* seems to be that proper evidence of the existence of a crime requires more than mere *discursive* evidence—that is, more than merely a *coherent story*.

The reason? Sheer discursive skill can produce a compelling story.

To get beyond mere coherence, there needs to be independent corroboration of the charge.

The assumption is that there is, or at least can be, a relevant "independence" between what can be produced by way of tangible, concrete evidence and the evidence represented by a coherent story.

Rational confidence is increased when independent sources corroborate each other.

Construction & Corpus Delicti: A Parallel?

Questions:

Is there room for a parallel distinction in mathematics?

More specifically, (i) is there a type of evidence in mathematics that comports reasonably well with the traditional conception of construction or exhibition, and (ii) is there a discernible relationship of "independence" between this type of evidence and other significant types of mathematical evidence?

Who Cares?

Some would say . . .

Gauss was surely right to think that there is no significant difference between the imaginary and other numbers as regards their 'reality.'

Whether his reasoning was altogether correct is less important. The point is that the real/ideal distinction is dubious and ought to be abandoned.

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Who Cares?

I disagree. . . for reasons I see as implicit in Euler's remarks on $\sqrt{-1}$.

I find in Euler the seeds of an alternative to the traditional understanding of the real/ideal distinction.

Instead of constructing it along the lines of a concept/intuition distinction, I think we should pay closer attention to Euler's remark concerning the "very natures" of the items involved.

Euler described the square roots of negative numbers as "numbers which *from their very nature* are impossible." He described them in this way because they violated the law of Trichotomy.

Euler's Criterion

This suggests a idea for thinking of the real/ideal distinction, an idea I'll refer to as *Euler's Criterion* for "ideality".

Euler's Criterion

An item is ideal to the extent that it violates basic laws of its relevant type.

Euler's Criterion thus suggests that the key test for ideality is whether an item departs radically enough from other objects of its supposed type. If it does, it's ideal (or more ideal). If not, it's real (or more real).

Euler's Criterion

It's worth considering, then, how the departure from basic laws of quantity in the case of the complexes might compare with the departures represented by other stages of the successive extension of the number-concept.

So let's look at what changes as we move from the natural numbers (positive integers) to the non-negative integers to the full set of integers to the rationals to the reals.

The following is a "conventional" description of these changes:

The Successive Extension of the Number-Concept

1. Naturals $\Rightarrow$ Non-negatives:

   $\forall x \forall y (x - y \neq x)$ law for the naturals, not for non-negatives.

2. Non-negatives $\Rightarrow$ Integers:

   $\forall x (x + 1 \neq 0)$ law for non-negatives, not for full integers.

3. Integers $\Rightarrow$ Rationals:
   $\forall x (2x - 1 \neq 0)$ law for integers, not for rationals.

4. Rationals $\Rightarrow$ Reals:

   $\forall x (x^2 \neq 2)$ law for rationals, not for reals.

Another way of describing these transitions, of course, is to say that, at each stage, solutions were added to (polynomially representable) problems which had no solution at the preceding stages.

The Successive Extension of the Number-Concept

Let's now add the move from the reals to the complexes to the preceding map.

1. Naturals $\Rightarrow$ Non-negatives:

   $\forall x \forall y(x - y \neq x)$ law for the naturals, not for non-negatives.

2. Non-negatives $\Rightarrow$ Integers:

   $\forall x(x + 1 \neq 0)$ law for non-negatives, not for full integers.

3. Integers $\Rightarrow$ Rationals:
   $\forall x(2x - 1 \neq 0)$ law for integers, not for rationals.

4. Rationals $\Rightarrow$ Reals:

   $\forall x(x^2 \neq 2)$ law for rationals, not for reals.

5. Reals $\Rightarrow$ Complexes:

   $\forall x(x^2 \neq -1)$ law for reals, not for complexes.

Euler's Criterion Again

The change made in this last step of the progression seems different from the others.

Here it seems most accurate to say that the law $\forall x(x^2 \neq -1)$ failed not because of the quantities involved, but because of the natures of the *operations*. It offends against our understanding of the very nature of *multiplication* to allow negative squares.

It doesn't seem that the other stages engender similar offense. In all of them, the natures of the operations stay the same; we simply(?) extend the domains to which the operations are applied.

Moreover, in the move to the complexes, we lose not only the law noted (i.e. $\forall x(x^2 \neq -1)$), and with it, our understanding of multiplication, we also lose at least one of the two seemingly essential laws of quantity, Trichotomy and Product.

Euler's Criterion Again

It's essentially reasoning of this type, I think, that lies behind Euler's conception of the real/ideal distinction.

I THINK . . . but we need to be careful.

In this connection, two related lines of questioning seem particularly worth pursuing.

Line 1: Does the above-mentioned "conventional" description of the successive extension of the number concept give us an accurate picture of the types of laws that have to be given up at its various stages? Specifically, are there stages other than the move to the complexes where there are laws of comparable basicness to Trichotomy and Product that have to be given up?

We don't SEE laws that look like this in the "conventional" description, but maybe this is more an indication of the superficiality of the conventional description than of the true state of affairs. This, at any rate, is a possibility, and not an idle one.

Euler's Criterion Again

Line 1a: Is whatever salient difference there may be between the move to the complexes and all the other moves well-described by saying, as suggested above, that it for the first time forces us to change our conception of a basic operation rather than our idea of the domain to which these operations may be applied?

Is this is a clear distinction? Does it, for example, offend less against our conception of multiplication to permit multiplication with negative multipliers (as distinct from negative multiplicands) than to permit squaring to generated negative products? Do we understand better what it is to add a thing to itself -3 many times than to allow negative squares? Or, for that matter, is it any less an offense against our concept of subtraction to permit a larger amount to be subtracted from a smaller than to permit square numbers to be negative?

These are clearly difficult and demanding lines of questioning. At the same time, though, progress in determining the plausibility of Euler's Criterion would seem to require progress with answering them.

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