Ideals of Proof

Varieties of Completeness

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Kronecker's Criticism of Dedekind

In his 1886 paper "Über einige Anwendungen der Modulsysteme auf Elementare algebraische Fragen" (cf. *Werke* III, 156, note *), Kronecker expressed a general opposition to what he termed "*Dedekind*ean concept-constructions" (*Dedekind*'schen Begriffsbildungen).

His opposition reflected what he believed were breaches of proper clarity and security in the methods Dedekind used in his work on algebraic ideals and modular arithmetic.

Kronecker's Criticism

He objected, in particular, to Dedekind's use of modules with what he termed "indefinite" (unbestimmten) numbers of elements.

He described reasoning making use of such collections as "overstepping" (Überschreiten) the boundaries of real (eigentliche) algebra.

Any proper arithmetic-algebraic (arithmetisch-algebraischen) investigation, he went on to say, must avoid use of such collections.

Kronecker's Criticism

The work to which Kronecker was objecting included (parts of) Dedekind's study of modular arithmetic.

In this work, Dedekind focused on homomorphisms of the natural numbers, whose images are the congruence classes of modular arithmetic.

As he saw it, this was the right focus to take in an arithmetical investigation because it featured the essential feature of the natural numbers—namely, their "structure".

Dedekind's Response

Two years later, in *Was sind und was sollen die Zahlen?*, Dedekind responded to Kronecker.

He described this response as a defense of the mathematician's freedom—particularly, his freedom to create and introduce new concepts in mathematics.

Dedekind described the importance of this freedom as follows:

"...the greatest and most fruitful advances in mathematics ...have been made above all by the creation (Schöpfung) and introduction of new concepts which is made necessary by the frequent occurrence of complex phenomena that could be managed (beherrscht) only laboriously (mühselig) using the old concepts."

Ded[1888], preface

Dedekind's Response

Dedekind thus defended the freedom to create and use new concepts in mathematics in the name of efficiency or *economy* of thought.

At the same time, he characterized Kronecker's program to limit this freedom—his program to "arithmetize" mathematics—both as lacking new insight and as tedious.

". . . it appears as something self-evident and not at all new that every theorem of algebra and higher analysis . . . can be expressed as a theorem about natural numbers—a claim I have heard repeatedly from the lips of Dirichlet. But I see nothing meritorious (Verdienstliches)—nor was this any part of Dirichlet's thinking—in actually performing this wearisome circumscription (mühselige Umschreibung) and demanding the use of nothing but the natural numbers."

Ded[1888], preface

Dedekind's Response: Footnote

Dedekind's defense might have been thought to require an examination and analysis of Kronecker's argument.

Dedekind disagreed, saying there wasn't any argument to examine.

"…not long ago Kronecker tried to impose certain restrictions on the free construction of concepts (der freien Begriffsbildung) in mathematics, restrictions I do not believe are justified; but there is no reason to go into more detail in this matter because the distinguished mathematician has yet to publish his reasons for the necessity or even the usefulness (Zweckmäßigkeit) of these restrictions."

Ded[1888], §1.2, note *

Dedekind's Plan

So, instead of rebutting attacks against free concept-construction in mathematics, Dedekind gave a direct defense of it.

His defense had two main parts.

Part I

- Extension of Creativism to the Natural Numbers: Argument that, analyzed at proper depth, the arithmetic of the natural numbers can be seen to be based on acts of concept-creation.

  (Thus, even the natural numbers that Kronecker so exhalted were based on acts of concept-creation.)

Dedekind's Plan

Part II

- Enlightened Conception of Objectivity: Proposal of new standard of exhibition, and argument that the concept-creations that lie at the base of arithmetic (i.e. the concept-creations involved in our most basic associations of things with things) support this type of exhibition with regard to the true "object" of arithmetic , namely, the *structure* of the natural numbers.

  (The concept-constructions upon which our number-concept is ultimately founded thus sustain something akin to what the traditional standard for "reality" or objectivity would require of it—namely, the construction or exhibition of an object "corresponding to" it.)

Thesis

Each of these parts of Dedekind's Plan was based on an ideal of completeness.

Creativist Part: Dedekind set the standard for analysis of "proper depth" by an ideal of completeness I'll call Probative Completeness. By enforcing this ideal, he would drive the analysis of the number-concept all the back to his logicist basis—the basic laws concerning our practice of making associations between things.

Objectivist Part: Employed the more familiar completeness ideal known as monomorphism or categoricity.

Probative Completeness

Dedekind presented the ideal of Probative Completeness in the first sentence of his essay.

"In science, what is provable should not be believed without proof."

("Was beweisbar ist, soll in der Wissenschaft nicht ohne Beweis geglaubt werden.")

He then added:

"But though this demand (Forderung) seems reasonable (einleuchtend), it does not seem to me to have been fulfilled even in the newest proposals for founding the simplest science; namely, that part of logic which deals with the theory of numbers."

Proposals cited were: Schröder's *Lehrbuch der Arithmetik und Algebra* (1873), Kronecker's "Über den Zahlbegriff" (1887) and Helmholtz' "Zählen und Messen" (1887).

Dedekind's Theses

Dedekind thus offered two theses:

- Positive Thesis: In science, proper justification of belief requires proof whenever proof is available.
- Negative Thesis: This requirement has not been met by even the latest attempts to found even the simplest science—namely, that part of logic that deals with arithmetic.

Our focus will be the Positive Thesis.

Dedekind's Defense of Probative Completeness as an Ideal

Dedekind's assertion of the ideal of Probative Completeness was not a case of a mathematician making foundational small-talk. He was very serious about it.

Just how serious can be seen from the elaborate argument he gave for it.

"... many a reader ... will be frightened by the long series of simple inferences corresponding to our step-by-step understanding ... and will become impatient at being compelled to follow proofs for truths which to his supposed inner intuition (innere Anschauung) seem at once evident and certain. ... it is exactly in this possibility of reducing such truths to others more simple, no matter how long and apparently artificial the series of inferences, that I recognize a convincing proof that ... belief in them is never given by inner intuition but is always gained only by a more or less complete repetition of the individual inferences."

Ded[1888], preface

Dedekind's Defense of Probative Completeness as an Ideal

He continued ...

"I like to compare this action of thought, so difficult to trace on account of the rapidity of its performance, with the action which an accomplished reader performs in reading; this reading always remains a more or less complete repetition of the individual steps which the beginner has to take in his wearisome spelling out ... Likewise from the time of birth, continually and in increasing measure, we are led to relate things to things and thus to use the faculty of the mind on which the creation of numbers depends. By the continuing recurrence of this practice ... we acquire a store of real arithmetic truths to which our first teachers later refer as something simple (Einfaches), self-evident (Selbstverständliches) and given in inner intuition (inneren Anschauung). Thus it happens that many very complicated notions (as for example that of the number (Anzahl) of things) are erroneously regarded as simple."

Ded[1888], preface

Dedekind's Daring Argument

There's a lot going on in these remarks, but the essentials seem to be as follows:

- Though it may involve long and seemingly artificial chains of inference to do so, it is *possible* to derive our ordinary arithmetical beliefs from simpler judgments concerning the results of rudimentary acts of association.

  $\therefore$

- Our ordinary arithmetical beliefs are, *as a matter of fact*, never based on intuition (so so-called "self-evidence" typically disguises large composites of reasoning) but always on more or less complete recapitulations of long inferential chains that lead back to judgments concerning rudimentary acts of association.

  Furthermore

- These beliefs *should* be so based (Ideal of Probative Completeness).

Dedekind's Daring Argument

This reasoning does not immediately inspire confidence.

It takes us first from

- It's *possible* to justify our ordinary arithmetic beliefs in a certain way

  to

- Our ordinary arithmetical beliefs *actually are* justified in this way.

Then, as if this wasn't bad enough, it takes us further to

- Our arithmetical beliefs *ought* to be justified in this way,

Doesn't look good, but we need to be careful. More later ...

Weyl's Objection

In *Das Kontinuum* (1918), Weyl stated an objection to Probative Completeness as an ideal of mathematical knowledge.

"In the preface to Dedekind (1888) we read, "In science, whatever is provable should not be believed without proof." This remark is certainly characteristic of how most mathematicians think. Nevertheless, it is a preposterous principle. As if such an indirect concatenation of grounds, call it a proof though we may, might awaken any "belief" not based on our assuring ourselves, through immediate insight, that each individual step is correct! In all cases, it is this process of confirmation (and not the proof) that remains the ultimate source from which knowledge derives its authority; it is the "experience of truth" (Erlebnis der Wahrheit)."

Weyl [1918], fn 19 (ch. 1)

Weyl's Objection

In Weyl's view then, Dedekind, through oversight or blindness, had emphasized the wrong thing.

- Oversight or blindness?
  - Dedekind was too fixated on proof as a source of arithmetical knowledge.
  - He thus lost sight of the fact that the ur-sources of our arithmetical knowledge are those *primitive intuitive realizations of truth* (Erlebnisse der Wahrheit) that form the building blocks of proof.
  - Had he had this in proper focus, he would not have emphasized proof as a source of knowledge in the way that he did.

Weyl's Objection

In Weyl's view then, Dedekind's advocacy of probative completeness overlooked the fundamental role played by intuition in the development of our arithmetical knowedge.

Indeed, as Weyl saw it, intuition is not only justificatively but developmentally basic to our arithmetical knowledge.

Hence his talk of belief's needing "the experience of truth" in order to be "awakened."

Weyl thus judged Dedekind to be wrong on both his main points—wrong about how our arithmetic knowledge should be developed, and wrong about how it actually is developed.

Weyl vs. Dedekind

Weyl seems right to note the indispensable role of non-inferred judgments in the larger development of our knowledge.

On the other hand, Dedekind was right to have noted that much of what seems immediate and self-evident to us is actually the result of massive inferential processing.

I would also note, contrary to what Weyl may seem to suggest, that he did not hold the absurd view that all our beliefs, all the way down, are based on inference. That *would* have been a preposterous view. But there's no indication Dedekind held it.

Weyl vs. Dedekind

What Dedekind did claim is that the beliefs our "first teachers" refer to as "self-evident" and "given in inner intuition" really are not. They're rather the products of elaborate inferences repeated so often and so rapidly as to be undetectable.

Dedekind was emphasizing that "evident but without readily perceivable inferential justification" should not be confused with "self-evident" or "given in inner intuition." He was right to do so.

He was also right to warn us that what may often pass for "self-evident" is really only "evident but without readily perceivable inferential justification."

This being so, we should not allow appearances of self-evidence to deflect us from the important task of determining what may be behind them.

Dedekind et al.

Seen this way, the ideal of Probative Completeness is an injunction to search for grounds.

It reminds us of Frege's memorable call to look for what it is that holds our arithmetic beliefs so firmly in place.

"The aim of proof is ... not merely to place the truth of a proposition beyond all doubt, but also to afford us insight into the dependence of truths one upon another. After we have convinced ourselves that a boulder is immovable ... there remains the further question, what is it that holds it so firmly in place?"

Frege, *Grundlagen*, §2

Dedekind et al.

It reminds us still more of Bolzano's more specific claim that the self-evidence of a proposition *does not absolve us of the obligation* to search for a proof of it.

"I stipulate the rule that the obviousness (Evidenz) of a proposition does not absolve us from the obligation to look for a proof of it, at least until I clearly realize why absolutely no proof could ever be given.†"

Bolzano, *Considerations on some Objects of Elementary Geometry* (1804), preface

Dedekind's Daring Argument

With these ideas in mind, we can smooth one of the jolting inferences of Dedekind's argument—the inference from

- Our ordinary arithmetical beliefs are justified by proof,

  to

- Our ordinary arithmetical beliefs *should be* justified by proof.

This inference is just Dedekind's way of saying that (i) we should look for the reasoning that often lies behind appearances of self-evidence, and that (ii) when we do this with our ordinary arithmetical beliefs, there will be a proof to find.

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Dedekind's Daring Argument

We can also lessen the abruptness of the inference from

- It's *possible* to find proofs that justify our ordinary arithmetical beliefs,

  to

- Our ordinary arithmetical beliefs actually are justified by proofs.

If, like Dedekind, we actually trace what we think are psychologically realistic genealogies for these beliefs, our geneaologies will have some plausibility as an actual account of our own acquisitions.

Right or wrong, Dedekind seemed to take this view of his derivations.[†] Given that he did, the inference above was not an unreasonable one for him to make. The real premise wasn't merely a claim of possibility, but a claim to know a psychologically realistic way of justifying our ordinary arithmetical beliefs by proofs.

†: Here we see clear divergence between Dedekind and Frege. Frege wanted to know what makes our arithmetic beliefs *true*, not what more basic beliefs these beliefs may rest on.

Dedekind's Argument: Conclusion

Dedekind's plan to show that even the arithmetic of the natural numbers was based on the creation of new concepts was based on his ideal of Probative Completeness.

His defense of that ideal is not as precarious as it might at first sight seem.

Other Responses to Probative Completeness

Others were less critical of Dedekind's ideal.

The American mathematician G. B. Halsted viewed it as intended to insure elegance or simplicity in the choice of axioms.

Specifically, it's aim was to reduce the axioms to the smallest number possible.

Roughly, the idea is that the more we push our beliefs into the domain of the inferentially derived, the fewer we must retain as axioms.

Axiomatic Completeness

This raises an interesting question connected with Weyl's response—namely, whether such elegance or simplicity should be pursued.

Weyl's view suggests maybe it shouldn't be.

If intuitive justification represents what is in some important way(s) the ultimate in justification, then should we try to limit the number of its occurrences in mathematics, or should we try to maximize it?

This suggests an alternative ideal of completeness that I'll call *Axiomatic Completeness*.

If a proposition is genuinely self-evident, it should not be believed on the basis of proof.

Axiomatic Completeness

Dedekind may have been right to question our impressions of self-evidence.

This does not mean, though, that there aren't genuinely self-evident beliefs.

Nor need it be taken to deny that genuine self-evidence isn't a type of justification that's especially valuable.

Axiomatic Completeness

If this is right, then we have not only Probative Completeness to consider, but Axiomatic Completeness as well.

Understood rightly, these seem to be compatible ideals.

This notwithstanding, Axiomatic Completeness clashes with certain traditional ideals of proof in a way that Probative Completeness does not.

Here I'm thinking of the traditional ideal of independence.

If we really are committed to making as many genuinely self-evident propositions into axioms as we can, we may not also be able to guarantee their independence. There's nothing in the notion of genuine self-evidence to suggest that one genuinely self-evident proposition couldn't imply another.

We thus see interaction between Axiomatic Completeness and two familiar ideals of proof: (i) elegance, simplicity or minimality (call it what you will) of the axiom set, and (ii) independence.

Axiomatic Completeness: Interaction with other Ideals of Proof

In both cases, there is tension.

In the former case, any special justificative value of genuine self-evidence would bid us include as many genuinely self-evident propositions into a system as possible, and to include them *without* proof. This could clearly lead us to away from the ideal of minimality.

Similarly for independence. Including as many genuinely self-evident propositions in our axioms as possible might well compel us to include logically dependent axioms in our axiom-set.

I end, then, with a question . . .

Question: If conflicts arise between Axiomatic Completeness and minimality or Axiomatic Completeness and independence, how should they be resolved? Specifically, which of these ideals of proof is (are) the more compelling?

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