Ideals of Proof (IP)

Overview

The term 'ideals' in the title has two intended senses. The first concerns the aims and virtues of proof considered as a justificative norm for mathematical practice generally. The second concerns the use of so-called "ideal" elements or methods as instruments of proof.

Ideals in the first sense include not only such traditional standards as rigor, certainty, *apriority*, purity and explanatory gain, but also such "collective" or systemic virtues as (various types of) completeness, closure, efficiency and freedom. Generally speaking, we want to improve our understanding of why such conditions and constraints as have figured as ideals of proof in the history of mathematics have so figured and whether they are truly deserving of such regard.

Ideals in the second sense include such things as the introduction of "infinites" (both large and small), imaginary and complex numbers in algebra and analysis, the use of Kummer ideals in number theory and the use of points, lines and planes at infinity in projective geometry.

We're concerned with ideals in both of the above senses. We're also interested in the relationships between them and are especially concerned to determine how the use of ideals in the second sense may either support or run contrary to realization of the ideals in the first sense. More generally, we want to identify and understand the contributions ideal elements or methods have made and may yet make to the larger enterprises of mathematical proof and knowledge.

One example of the type of question with which we're concerned comes from the frequently encountered claim that the use of ideal elements stands to improve our efficiency as problem-solvers and theorem-provers. The following remark by Hadamard illustrates the view.

> ... the shortest and best way between two truths of the real domain often passes through the imaginary one.
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> "An Essay on the Psychology of Invention in the Mathematical Field" (1945)

Such claims raise many questions. Are they, for example, intended to suggest that the use of ideal elements in some sense affords gains (either of quality, extent or efficiency) in the attainment of mathematical knowledge? And does this further imply that solutions and theorems produced through application of ideal methods are to be taken as genuinely adding to our *knowledge*, and that they do so while consuming fewer resources than alternative methods?

Another example comes from the common claim that the distinction between real and ideal elements in mathematics is confused or implausible and ought to be abandoned. The following remark by the 20th century American mathematician James Pierpont is typical of this view.

> ... all numbers are equally real and equally imaginary. Historically ... the term imaginary still clings to the complex numbers; pedagogically we must deplore using a term which can only create confusion ...
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> *Functions of a complex variable* (1914)

Like Hadamard's remark, this remark too raises certain questions. Among these is that concerning whether the several stages of the successive extension of the number-concept are indeed alike or whether there are significant differences between them. Each stage of extension requires the abrogation of certain theorems. Are the theorems relinquished at one stage of the same basic character and centrality as those relinquished at others? If there are differences, how important are they, and what do they signify? If, for example, one stage of extension required relinquishment of more basic or important theorems than another, would this be evidence of a difference in the degree or character of the imaginariness or ideality between them?

These are but two illustrations of inquiries belonging to the *IP* project. Others concern (i) how the use of ideal elements/methods may affect the pursuit of rigor and such higher epistemic goals as purity and explanatory adequacy, (ii) what consequences admission of ideal elements/methods might have for such things as the role of constructive reasoning in proofs, (iii) what type(s) of freedom the use of ideal elements and methods represents and what place the exercise of such freedom(s) rightly has in proofs and other forms of mathematical reasoning, (iv) the relationship between the use of ideal elements and methods and various ideals of completeness or closure and (v) what effects admission of ideal methods in proof might have on the "logic" of the notions of mathematical proof and provability.

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IP is an interdisciplinary initiative intended to bring researchers from a variety of disciplines together to achieve a better understanding of that distinctive higher human cognitive function that is mathematics. We welcome inquiries and proposals from scientists and scholars of all ranks and disciplines who believe they have something to contribute to the project. More information may be obtained by clicking on the buttons for the various subprojects.

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