Subproject I: Ideal Elements and the Increase of Resolutory Capacity

(a) Imaginary & Complex Numbers

Viète famously claimed that the aim of algebra was to leave no problem unsolved. Three centuries later, Hilbert made a similar claim, saying that in mathematics there is no Ignorabimus. Every mathematical problem is solvable, he said, by "pure reason".

This is clearly an ambitious resolutory ideal. So much so, perhaps, as to be unattainable. This notwithstanding, resolutory optimism has been the rule in the history of mathematics and the use of ideal elements has been a prime sustainer of this optimism. This is illustrated by the role played by ideal elements in the systematic extension of the number-concept.

Descartes' use of analytic methods in geometry resulted in a standard of sorts for representing geometrical problems and their solutions. To put it roughly, the problems (i.e. problems calling for the construction of lines) were reduced to polynomial equations and their solutions to roots of these equations: a geometrical problem represented in this way thus had a solution if its representing polynomial had a root.[1]

Each succesive stage of the extension of the number-concept increased resolutory capacity. Thus, for example, the addition of negative numbers to the positive numbers yields roots for polynomials of the form $x + n = 0$ (n a positive integer), hence solutions for problems that lack solutions in the positive integers. The addition of the complex numbers to the real numbers is an even more striking case of this type. For unlike the addition of 0, the negative integers, the rationals and the irrationals, the addition of the complex numbers produces for the first time an *algebraically closed* set—that is, a set that contains a root (solution) for every polynomial whose coefficients also come from that set. Every problem that is formulable in terms of the complexes is thus also solvable in terms of them. This being so, the extension to include the complexes comprises a case where the addition of

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[1] This is an oversimplification the refinement of which we'll not develop here. To mention but one complication, though, Descartes distinguished between real or genuine roots, on the one hand, and roots that were "solely imaginary (*seulement imaginaires*)" (cf. *The Geometry*, Smith & Latham (trans.), p. 175), on the other. Commenting on this distinction, Descartes remarked that though we can conceive of the imaginaries, "there is not always a definite quantity corresponding to" these conceptions (*loc. cit.*).

ideal elements dramatically increases resolutory capacity. Such, at any rate, is the usual view. There are, however, problems that must be attended to before such a view can be accepted or even evaluated. These include the following two.

Problem I.a.1

Are the solutions provided by the addition of the complex numbers really "solutions"? In asking this, we're not querying the correctness of the Fundamental Theorem of Algebra. Rather, we're asking for clarification on another point, namely, the *quality* of the solutions that are provided for by the addition of the complex numbers. The question has force because extending to the complexes forces abrogation of certain theorems that appear to be important general laws of quantity. In particular, we have to give up at least one of the following two laws.

(A): For every x, either $x < 0$ or $x = 0$ or $x > 0$.

(B): For every x, y, if $x, y > 0$, then $x + y > 0$ and $xy > 0$.

The centrality of (A) and (B) as laws of quantity should cause us to ask whether entities that don't satisfy them can righly be seen as quantities at all and, hence, as genuine "solutions" to problems concerning quantities. (Euler expressly raised this question with regard to (A). For him, any entity that was "neither nothing, nor greater than nothing, nor less than nothing" did not qualify as a possible quantity.) To properly investigate this question would require in the first place an historico-conceptual study to help clarify the concept of quantity that figured in the work of early algebraists.

A next natural step would then be to systematically compare the several stages of the extension of the number concept to determine how the theorems relinquished at each stage compare to those relinquished at the others in terms of their centrality as laws of number or quantity. On the face of it, there appear to be differences. When we add the rationals, for example, we have to give up the theorem that there does not exist a number which, multiplied by 2 gives 1. This doesnt appear to be a basic law of quantity, though. We can easily imagine cutting at least some quantities into two equal parts. We cannot so easily imagine a quantity that is neither nothing, nor less than nothing, nor greater than nothing. Ultimately, of course, what will be needed is a careful treatment of whether there is a way of making sense of talk of mathematical "laws", what purpose or purposes such talk/thinking serves and what clear sense could plausibly be given to claims to the effect that one theorem is more "basic" than another when considered as a "law" of number or quantity.

Problem I.a.2

In what sense does the addition of the complex numbers "provide for" the solution of problems? It provides for their existence of course. But does it do more than this? Does it make actually finding them easier? If so when (i.e. in what cases) does it do this, and in what sense(s) of "easier" and to what extent? Are there systematic effects that can be described?

Here too a natural place to start is with an investigation of historical cases (e.g. the use of Cardano's law to solve cubic equations). Claims that the complexes shorten or simplify things are common (cf. Lamberts claim that "The sign $\sqrt{-1}$ represents an unthinkable non-thing. And yet it can be used very well to find theorems", or Hadamard's claim that "the shortest way between two truths of the real domain often passes through the imaginary" or Veblen and Young's claim that it is "easy to abbreviate the proofs of theorems . . . by a free use of imaginary elements").

What is missing are careful clarifications of what such claims might or ought to mean and how, so understood, they might be defended. Does the use of complex numbers provide systematic ways of simplifying known proofs of theorems concerning the real domain? Does having access to a theorem such as the Fundamental Theorem of Algebra systematically simplify the finding of proofs of theorems concerning reals? If so, how and to what extent, and where? Here is a place at which our project interlinks with the theory of computation, proof theory and also with cognitive science and cognitive psychology. Accordingly, we hope to attract partners who work in these areas.

The idea that complex numbers are to be valued chiefly for their systematic simplifying effects on proofs and computations concerning real quantities suggests a radical alternative to the approach assumed in Problem I.1. Instead of focusing on whether use of complex numbers provides quantities as solutions to problems, it focuses instead on the overall instrumental advantages of such use. Pursuing this alongside the approach taken in Problem I.1 will help balance the investigation of Subproject I.

It may also connect it with the larger development of hypercomplex numbers. Extension of the number-concept to quantities of dimension greater than 2 (the complexes being viewed as quantities of dimension 2 in their representation as pairs of reals) requires substantial departures from the usual laws of algebra. Hamilton's quaternions (quantities of dimension 4) required giving up the commutativity of multiplication. Graves' and Cayley's octonions (quantities of dimension 8) the additional abandonment of associativity for multiplication. Sedenions (quantities of dimension 16) require that we give up even power associativity (i.e. the equality between $x(xy)$ and $(xx)y$).

Hamilton had an interesting take on this. He believed that conceiving of quantities as multi-dimensional entities diminished the seriousness of "giving up" standard laws of algebra. As he saw it, these laws were principally conceived as laws for one dimensional quantities or what he termed "single numbers." He believed that, conceived as a one dimensional entity, $\sqrt{-1}$ was "absurd." Conceived as a two dimensional entity (a "couple") it was not. Laws that may feature as central laws for one dimensional quantities do not necessarily qualify as central laws for higher-dimensional quantities. Or so Hamilton maintained. His arguments have not gotten the kind of careful treatment they deserve, however, and this is something our project aims to improve on.

It is also worth noting that our project comes into contact with a mathematical question at this point. Looking at the hypercomplex hierarchy, one sees the following phenomenon. At dimension 2, we must give up either condition (A) or condition (B) above. At dimension 4 we must give up commutativity of multiplication, at dimension 8 full associativity, and at dimension 16 power-associativity.

Problem I.a.3: (Associated Mathematical Problem) Is there a positive integer n such that at dimension 2^n (or before) distributivity and/or other basic algebraic laws must also be given up? More generally, is there an n such that at dimension 2^n virtually all algebraic laws or basic laws of quantity must be abandoned? (To provide a convincing answer to such questions will, of course, require some sort of account of what a law or basic law of quantity is.)