

Subproject I: Ideal Elements and the Increase of Resolatory Capacity

(b) Elements at Infinity & Duality in Projective Geometry

It has frequently been claimed that the use of ideal elements increases the “efficiency” of mathematical reasoning. By this it has generally been meant that appeal to ideal elements somehow shortens or simplifies proofs and problem-solutions without compromising their reliability or other epistemic virtues.

Sometimes these efficiencies seem striking, as in the case of the so-called “dualities” that are made possible by the introduction of elements at infinity in projective geometry. For example, introducing a point at infinity to serve as the point of intersection of pairs of parallel lines induces symmetries of the following type.

- i. Any two distinct points of a plane are on one and only one line.
- ii. Any two distinct lines of a plane are on one and only one point.

Here each of the sentences can be obtained from the other by interchanging the terms ‘point’ and ‘line’.

The distinguished twentieth century geometer H. S. M. Coxeter described the efficiencies represented by such dual pairs of theorems in the following way:

“One of the most attractive features of projective geometry is the symmetry and economy with which it is endowed by the principle of duality: fifty detailed proofs may suffice to establish as many as a hundred theorems.”

Such statements give the impression that the efficiency represented by the duality phenomena in question is some type of *epistemic efficiency*—that is, efficiency concerning the optimal utilization of (limited) epistemic resources. Duality is supposed to improve epistemic efficiency by providing a simple, mechanical means of transforming a proof of one theorem into a proof of another (its dual).

The idea is that we start with a proof of one theorem (call it the *primary* theorem), apply the simple and easy duality transform to it and obtain a proof of the dual (*secondary*) theorem. Since the duality transform is simple and easy, the idea is that applying it to find a proof of the dual theorem consumes far fewer epistemic resources than would finding a proof of the secondary theorem by more direct or *ex nihilo* means. The idea seems to be that with a slight additional expenditure of resources, dualization adds a second theorem to our knowledge.

There are difficulties concerning this thinking, however. To see what some of these are, let's consider more carefully the reasoning that stands behind it, reasoning Ill refer to as the *naive argument*. This reasoning is as follows:

1. A proof by dualization provides for knowledge of the secondary theorem it proves.
2. It therefore provides for an addition to content we know.
3. At the same time, it consumes fewer resources than would the direct or *ex nihilo* proof of the secondary theorem.

Therefore, on any reasonable definition of relative epistemic efficiency, it follows that

4. proof by dualization provides for improvement in epistemic efficiency when compared to direct or *ex nihilo* proof.

Does this argument provide a compelling defense for the claim that duality advances epistemic efficiency? We think not and now indicate but one of several problems one encounters in any attempt to defend the *naive argument*.

Problem I.b.1

This problem concerns premise 2. and the difficulty of determining whether (and if so, to what extent) dualization indeed adds to the content of what we know. A prime consideration in this connection is that some secondary theorems seem more nearly stipulative or conventional than others.

This is illustrated by the duality between i. and ii. above. On the ordinary interpretation, i. is true and ii. false. Interpreted this way, then adding ii. as a theorem could not be taken as adding to our knowledge. It becomes 'true', of course, when a so-called point at infinity is stipulated to serve as the point of intersection of pairs of parallel lines.

If this is done, however, the question then becomes whether, given this dependency on stipulation, ii. has the character we generally associate with genuine knowledge (at least in its more highly valued forms). One cannot generally add to her knowledge sheerly through stipulation. And though proposition ii. may not be sheer stipulation, it is somehow a product of it. The problem thus arises of finding a principled way of determining the conditions under which and the extent to which adding theorems by dualization truly corresponds to enlarging the content of our knowledge.