Subproject II: Ideal Elements and Construction

(a) Freedom, Rigor and Purity

The use of ideal elements has often been justified by appealing to a basic freedom the mathematician is taken to have—the freedom to create instruments. There are indications of this idea already in Leibniz, and many more explicit statements in the writings of 17th, 18th and 19th century mathematicians and philosophers. It remained a popular idea among 20th century thinkers too, with affirmations of one sort or another from such otherwise divergent foundational thinkers as Dedekind, Poincaré and Hilbert.

Poincaré, for example, affirmed the mathematician's freedom to create, but saw this freedom as extending mainly to the creation of symbols and symbolic systems. He thus wrote (*Science & Hypothesis*, 27) "the mind has the faculty of creating symbols, and it is thus that it has constructed the mathematical continuum . . . The only limit to its power is the necessity of avoiding all contradiction . . . ." He went on to add, though, that the mind only exercises this freedom when experiment or experience bids it do so.

Hilbert's conception of freedom was more liberal in certain respects. It allowed in the first place that the mathematician could freely create not only symbols but concepts. Secondly, he didn't believe the exercise of creative freedom to be dependent on experience in the way Poincaré suggested. Rather, he saw it as admitting of more imaginative applications when the need for efficient instruments arose. He believed, moreover, that use of the axiomatic method was perhaps the most important form that exercise of this freedom took.

Hilbert's general understanding of the mathematician's freedom was presented in his so-called "creative principle" (*schöpferische Prinzip*), which he stated as follows:

> "Having arrived at a certain point in the development of a theory, I may designate (*bezeichnen*) a further proposition as correct (*richtig*) as soon as it is recognized (*erkannt*) that its introduction results in no contradiction with propositions previously admitted as correct . . . ."

One curiousity concerning this way of understanding the mathematician's freedom is the way in which it contrasts with older understandings of freedom and the role they were taken to play in mathematics. One important conception of this role was stated and defended by Bacon and Hobbes in modern times. Hobbes put the point well in his *Six Lessons*:

> "Of Arts, some are demonstrable, others indemonstrable; and demonstrable are those the construction of the Subject whereof is in the power of the Artist himself; who in his demonstration does no more but deduce the Consequences of his own operation. The reason whereof is this, that the Science of every Subject is derived from a praecognition of the Causes, Generation, and Construction of the same; and consequently where the Causes are known, there is place for Demonstration; but not where the Causes are to seek for. Geometry therefore is demonstrable; for the Lines and Figures from which we reason are drawn and described by our selves . . . ."

For Hobbes, then, as for a variety of other thinkers (particularly those working within the so-called *Maker's Knowledge* tradition), the capital mathematical freedom was a freedom to create—specifically, to effectively cause or construct—figures. The creator's constructional operations were the causes both of the objects constructed and of their properties. They were also, in Hobbes' view, transparent to the creator and gave her the type of knowledge that was needed for genuine proof or demonstration—namely, knowledge of cause.

Poincaré rightly saw indications of the same basic thinking—namely, as emphasis on keeping "causes" before the prover's mind—in Kronecker. He thus wrote: "Kronecker is above all concerned that the philosophical meaning of mathematical truths be put in evidence. Whole numbers being the foundation of everything, he wants them to remain in view throughout. For him, only addition and multiplication are legitimate operations. It is only as a concession to contemporary prejudices that he even admits division."

Along with this view of proof went a *genetic* conception of rigor. A rigorous proof was taken to be one in which the subject of the proof was kept present before the prover's mind throughout the course of the proof. This *presentist* conception of rigor was common from the late seventeenth to the late nineteenth century and was often presented as a reason to prefer the constructional methods of classical geometry to the abstract symbolical methods of the then-modern algebra. As MacLaurin put it:

> "In Geometry the Representations are more Natural, in Algebra more Arbitrary . . . The former are like the first Attempts towards the Expression of Objects . . . by drawing their Resemblances; the latter correspond more to the present Use of Languages and Writing. Thus the Evidence of Geometry is sometimes more simple and obvious."

We thus see two quite different conceptions of the role and significance of freedom in mathematics. According to the one (e.g. that advocated by Hilbert), freedom is a liberty to make tools to suit ones needs and interests, a freedom that is limited only by the need to remain consistent and to fulfill the purposes for which the tools created were originally wanted.[1] According to the other—that advocated by Hobbes and Kronecker—the mathematician's freedom is essentially a liberty to construct things, a liberty the exercise of which gives the creator special knowledge of the genesis or "cause", hence the nature, of what she creates.

One of the primary challenges facing the history and philosophy of mathematics today is to explain the dynamic that lead from the traditional preference for constructive methods to the recent upsurge in preference for the axiomatic method (in Hilberts sense) in the late 19th and early 20th centuries. To meet this challenge a number of problems must be dealt with. We mention but one here.

<u>Problem II.1</u>

There are competing conceptions of rigor that underlie the intuitionist and modern axiomatic (in Hilbert's sense) approaches to proof. Intuitionists (or at least some of them, including Brouwer) seem to favor a presentist model which sees proofs as having a subject and sees rigor as consisting in a continuous retention of this subject before the prover's mind throughout the course of the proof. Proofs must thus be "pure" (i.e. consist wholly of judgements about the proof's subject), and the transitions from one step to another must amount to subject-preserving transformations of one intuitive judgement into another. Axiomatists, on the other hand, favor a view of rigor like Pasch's which requires that the process of inferring be independent not only of all diagrams and similar "intuitions", but of the very senses of the terms involved. As he put it:

> "...if it is necessary to so think, the defectiveness of the deduction and the inadequacy of the method of proof is thereby revealed."

A thorough logico-historico-philosophical study of the strengths and weaknesses of these different conceptions of rigor (and their underlying conceptions of inference and proof) as well as their founding motives has yet to be given. It is therefore an appropriate topic for us to investigate.

[1]Hilbert generally referred to this latter type of condition as *fruitfulness*.