Subproject III: Ideals of Completeness:

(a) Dedekind's Ideal

The opening words of the preface to the first edition of *Was sind und was sollen die Zahlen?* (1888) presented an intriguing ideal of completeness for mathematical reasoning—an ideal Dedekind believed had not been met by even the simplest science, the science of arithmetic. He thus wrote:

> In science nothing that is provable ought to be believed without proof. But though this demand seems reasonable, it does not seem to me to have been fulfilled even in the newest methods of founding the simplest science, namely, that part of logic which deals with the theory of numbers.[1]
>
> Dedekind, *Gesammelte mathematische Werke*, vol. 3, 335

For Dedekind, then, the following seems to have been an ideal of any science $\mathcal{I}$.

***Dedekind Completeness:* For every proposition π pertaining to $\mathcal{I}$, if π is capable of proof, then π is provable in $\mathcal{I}$.**

Dedekind Completeness is interesting not least because of its differences with other, more familiar completeness ideals. A case in point is the modern ideal that all *truths* pertaining to $\mathcal{I}$ should be provable in $\mathcal{I}$. Dedekind Completeness does not imply this, and to secure such implication would require showing that every true proposition pertaining to $\mathcal{I}$ is capable of proof—something that Dedekind seems not to have maintained and something which, in any event, is not evident.

[1] The German is:

> Was beweisbar ist, soll in der Wissenschaft nicht ohne Beweis geglaubt werden. So einleuchtend diese Forderung erscheint, so ist sie doch, wie ich glaube, selbst bei der Begründung der einfachsten Wissenschaft, nämlich desjenigen Teiles der Logik, welcher die Lehre von den Zahlen behandelt, auch nach den neuesten Darstellungen noch keineswegs erfüllt anzusehen.

At the heart of Dedekind's ideal is the notion of a proposition's being prov*able*, or being "capable" of proof. The notion of proof that figures here seems clearly to be a normative one. What Dedekind is suggesting is that for a π that's capable of proof, there's something epistemically deficient about belief in π that's not based on such proof. For π that are capable of proof, then, one *ought* not (or cannot adequately or optimally) to *believe* π except on the strength of a proof.

For Dedekind, epistemic or justificative force was intrinsic to proof. It has, by its very nature, justificative force or potential.

Dedekind Completeness and the thinking behind it raise a variety of interesting questions. These include, but are not limited to, the following.

I. What is it for a proposition to be *capable* of proof? What does *being capable of proof* essentially consist in?

    i. It may be that not every proposition that is not capable of proof is a proposition that needs proof. How ought we to understand the relationship between propositions that are *capable* of proof and those that *require* proof (if, say, they're to be rationally, or in some sense "optimally" believed)?

    ii. In particular, does the appeal of Dedekind Completeness depend on every proposition's being either capable of proof or not needing it? Can Dedekind Completeness have force as an ideal of proof without invoking such a precondition?

    iii. Can propositions that are evident be capable of proof? If so, does this imply that proof must do more than make its conclusion evident?

II. What type of failure is it not to provide a proof for a proposition that is *capable* of proof?

    i. How, if at all, might it differ from failure to provide a proof for a *true* proposition?

    ii. Can evident propositions can be capable of proof? Can self-evident propositions, or even self-evidently certain propositions, be capable of proof? If so, what does this tell us about what proof provides or can provide?

III. Did Dedekind believe that evident or self-evident propositions can be capable of proof?

    i. What was the connection between Dedekind's logicist convictions and his understanding of what it means to say that a proposition is capable of proof?

    ii. What if any bearing did his belief that "the number-concept [is] entirely independent of the notions or intuitions of space and time ... [and is] an immediate product of the pure laws of thought. . . . numbers are free creations of the human mind" have on his understanding of what it means for a proposition to be capable of proof? What bearing might or ought it to have had?