

Subproject IV: Ideal Elements and Ideals of Efficiency

(c) Efficiency & the Evaluation of Hilbert's Program

Efficiency considerations appear to have important ramifications for the proper understanding and evaluation of Hilbert's Program. Generally speaking, Hilbert prized the use of ideal elements for the efficiency or convenience it represented. Somewhat more accurately, he believed that proofs making use of ideal elements/methods were sometimes, in fact often, easier to *discover* than proofs exclusively employing the contentual methods of finitary reasoning. This suggests that the uses of ideal elements/methods that Hilbert really wanted to preserve were those that represent some gain in efficiency as compared to their simplest or most efficient finitary counterparts.

For ease of expression, let's introduce some terminology. Call an ideal proof π_I of a real theorem ρ gainful if π_I is easier to discover than any purely real proof of ρ . Further, for a given system \mathcal{I} that formalizes a body of ideal reasoning let \mathcal{I}_{gain} be the set of theorems that results from removing from \mathcal{I} all real theorems of \mathcal{I} for which there is no gainful ideal proof in \mathcal{I} .¹ I believe there is an historically and philosophically plausible interpretation of Hilbert according to which his fundamental commitment was to defend not the ideal methods *per se*, but the *gainful* ideal methods.

This raises questions concerning the traditional evaluation of Hilbert's Program. The reasoning behind that evaluation characteristically relies on a premise to the effect that what Hilbert classified as *real* (or finitary) reasoning is a subsystem of the ideal reasoning to the proof of whose consistency Hilbert was properly committed. This premise, however, is open to doubt, even for someone who may accept that all finitary reasoning is formalizable in \mathcal{I} . For in those cases where \mathcal{I}_{gain} is a subset of the theorems of \mathcal{I} , it may be that \mathcal{I}_{gain} is axiomatizable by a set of axioms weaker than the axioms of \mathcal{I} . This being so, it cannot reasonably be maintained, without further argument, that the consistency of \mathcal{I}_{gain} is finitarily provable only if the consistency of \mathcal{I} is.

This in itself, of course, gives no positive reason to believe that Hilbert's Program can be carried out, either in whole or in significant part. It does, however, suggest that the

¹Note that we characterize \mathcal{I}_{gain} as a set of theorems, not as a formal axiomatic system. Note further that from the assumption that \mathcal{I}_{gain} is a proper subset of the theorems of \mathcal{I} , it does not follow that the axioms of \mathcal{I} that are retained in \mathcal{I}_{gain} is a proper subset of the axioms of \mathcal{I} .

ultimate evaluation of Hilbert's Program is more subtle and complex than it has generally been taken to be. The following questions are indicative of some of this complexity.

Problems IV.c.1– 4

1. What are the conditions that a concept of efficiency of the sort Hilbert had in mind—what was referred to above as *discovermental* complexity—ought to satisfy? How might such efficiency be reckoned or measured?
2. Are there significant ideal systems \mathcal{I} , and significant families \mathcal{R} of real theorems such that for every ρ in \mathcal{R} , there is at least one real proof that is as efficient (however efficiency is measured) as any ideal proof of ρ in \mathcal{I} ? Are there \mathcal{I} of this type where the consistency of $\mathcal{I} - \mathcal{R}$ is finitarily provable?²
3. Are there significant ideal systems \mathcal{I} and irredundant ideal axioms γ of \mathcal{I} such that γ is not used in any gainful proof in \mathcal{I} of any real theorem? Are there systems of this type where the consistency of $\mathcal{I} - \Gamma$ ³ is finitarily provable?
4. Are there significant ideal systems \mathcal{I} and irredundant ideal axioms γ of \mathcal{I} such that every real theorem in whose gainful proof γ participates has equally gainful proofs not involving γ ? Are there systems of this type where the consistency of $\mathcal{I} - \Gamma$ is finitarily provable?

²Some of the difficulties here, of course, concern what $\mathcal{I} - \mathcal{R}$ might or ought to be taken to mean. Similarly for the systems $\mathcal{I} - \Gamma$ mentioned below.

³Here and in 4. Γ stands for the set of all γ of the type mentioned.